Revista Mexicana de Astronomía y Astrofísica, 61, 1-10, (2025)

OPEN ACCESS https://doi.org/10.22201/ia.01851101p.2025.61.02.01

RMxAA REVISTA MEXICANA DE ASTRONOMÍA Y ASTROFÍSICA

An Oscillating Rastall Universe Crossing the Phantom Divide Line

Nasr Ahmed[1,2], **Anirudh Pradhan**[3] **and Archana Dixit**[4]

[1]*Mathematics Department, Faculty of Science, Taibah University, Saudi Arabia.*
[2]*Astronomy Department, National Research Institute of Astronomy and Geophysics, Helwan, Cairo, Egypt.*
[3]*Centre for Cosmology, Astrophysics and Space Science (CCASS), GLA University, Mathura-281 406, Uttar Pradesh, India.*
[4]*Department of Mathematics, Gurugram University, Gurugram-122003, Haryana, India.*

Keywords: *Causality, Cyclic universe, Modified gravity, PACS number: 04.50+h, 04.20.Jb, 95.35+d, 98.80.Es, Phantom crossing*

Abstract

A cyclic flat universe with Quintom behavior and a future Big Rip as presented in the framework of Rastall gravity, which is an extension of the standard ΛCDM model. The Hubble parameter oscillates periodically between positive and negative values in each cycle. Cosmic transit has been simulated through an oscillating time-dependent deceleration parameter and is expected to occur at approximately 8.7 Gyr. Causality is satisfied at all times, except near the initial singularity and the future Big Rip singularity. The apparent horizon, entropy, and other thermodynamic quantities associated with the current model were analyzed. The energy conditions were investigated.

Resumen

Se presenta un modelo de universo plano y cíclico con comportamiento *Quintom* y *Big Rip* o gran desgarramiento futuro, en el contexto de gravedad tipo Rastall. El modelo es una extensión del modelo estándar ΛCDM. El parámetro de Hubble oscila periódicamente de un ciclo a otro entre valores positivos y negativos. El tránsito cósmico se ha simulado con un parámetro de desaceleración oscilante en el tiempo y se espera que ocurra en aproximadamente 8.7 Gyr. Se satisface la causalidad todo el tiempo, excepto en las singularidades inicial y la futura del *Big Rip*. Se analizan cantidades termodinámicas asociadas al modelo como horizonte aparente y entropía, entre otros. Se han investigado las condiciones energéticas.

Corresponding author: Nasr Ahmed *E-mail address:* abualansar@gmail.com
Received: February 11, 2025 **Accepted:** March 17, 2025

1. Introduction and Motivation

One of the most significant and fascinating phenomena in contemporary cosmology is the accelerating expansion of the universe. Our understanding of the composition and development of the cosmos underwent a paradigm shift when it was discovered in the late 1990s through the observation of Type Ia Supernovae (Perlmutter et al., 1999; Riess et al., 1998). Numerous independent observations, including large-scale structural surveys and measurements of cosmic microwave background radiation, support this groundbreaking discovery (Spergel et al., 2003a; Komatsu et al., 2009; Astier et al., 2006). At both large scales and strong curvature, the Einstein theory of General Relativity (GR) is inconsistent with observations. Cosmic acceleration has been a basic motivation behind modified gravity theories since it cannot be explained by GR without assuming dark energy (Riess et al., 1998; Copeland et al., 2006). Examples of such modified gravity theories are $f(R)$ gravity (Nojiri & Odintsov, 2006), Gauss-Bonnet gravity (Nojiri et al., 2008; Ahmed et al., 2023) where the Gauss-Bonnet term $\mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\delta}R_{\mu\nu\rho\delta}$ has been used in the action instead of the Ricci scalar R, and f(T) gravity (Ferraro & Fiorini, 2007) where the torsion scalar (T) has been used; the generalized $f(R,T)$ gravity, where T is the trace of the energy-momentum tensor. Additional examples include $\kappa(R,T)$ gravity (Ahmed & Pradhan, 2022) and extra-dimensional gravity theories (Overduin & Wesson, 1997; Sotiriou, 2011; Ahmed et al., 2020; Ahmed & Rafat, 2018).

Bouncing scenarios (also known as cyclic cosmology) were initially proposed as an alternative method to solve some of the problems of the standard model of cosmology, such as flatness and initial singularity. The universe in the singularity-free Big Bounce emerges from a previous contracting stage (Novello & Bergliaffa, 2008; Ijjas & Steinhardt, 2018; Sahoo et al., 2020; Nojiri et al., 2019), where the contraction-expansion cycle is assumed to continue forever. Cyclic cosmological models have been extensively studied through numerous modified gravity theories (Sahoo et al., 2020; Nojiri et al., 2019; Bamba et al., 2014a,b, 2015; Tripathy et al., 2019; de la Cruz-Dombriz et al., 2018; Ahmed et al., 2022; Singh & Shukla, 2020). A general cyclic model in $f(R)$ gravity was constructed by Pavlovic & Sossich (2017). A unitary version of Conformal Cyclic Cosmology (CCC) was suggested by Frampton (2018). An M-theory-inspired cyclic model with two branes was presented by Steinhardt & Turok (2002), where the EoS parameter satisfies $\omega \gg 1$ during the contraction phase. A new cyclic theory of the universe was presented in Ijjas & Steinhardt (2019), where the scale factor increased exponentially from one cycle to the next. This recent cyclic model resolves many problems and generates a nearly scale-invariant spectrum of density perturbations. An interesting singularity-free oscillatory model was discussed by Sharma et al. (2023), where the late-time evolution was probed in the context of quasi-steady-state cosmology (QSSC) (Hoyle et al., 1993, 1994) to alleviate the Hubble tension. The cosmic scale factor parametrization has been chosen according to the description in (Hoyle et al., 1993, 1994, 1995; Sachs et al., 1996) as

$$a(t) = e^{t/P} \left[1 + \eta \cos(2\pi t/Q)\right]. \tag{1}$$

Here, η is a dimensionless constant, and P and Q have dimensions of time. In the limit when the dimensionless constant η approaches zero, a de Sitter-like evolution occurs. As time goes to zero, the scale factor tends to the value $1+\eta$, where no singularity exists.

Rastall gravity is one such modification, proposed by Peter Rastall in 1972 (Rastall, 1972). Rastall gravity (RG) is a modified theory of gravity that features a non-conserved stress-energy tensor and a peculiar non-minimal coupling between geometry and matter. This has become an interesting and important theory of gravity in modern times. The conservation law that demonstrates a divergence-free energy momentum tensor in general relativity (GR) has been accepted; that is, $\nabla_\nu T^{\mu\nu} = 0$, where ∇_ν is the covariant derivative. Rastall adopted a new and distinct conservation law that conjectured GR concepts. In this theory, the energy momentum tensor does not a conserved quantity. Rastall formulation is defined as $\nabla_\mu T^\mu_\nu = \left(\frac{\kappa}{16\pi}\right) \nabla_\nu R$, where κ is the coupling constant and R is the Ricci scalar. The second Bianchi identity in this theory remains the same i.e., $\nabla^\mu G_{\nu\mu} = 0$, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ be the Einstein tensor. A notable feature of RG is that, whereas the mathematical component of this theory remains invariant, modifications have been applied exclusively to the matter component (De Moraes & Santos, 2019; Majernik & Richterek, 2006; Rastall, 1976). For the Rastall gravity framework, Visser (Visser, 2018) claimed that Rastall gravity may be completely equivalent to general relativity. In contrast, Darabi et al. (2018) have argued that Rastall gravity can be interpreted as a modified theory of gravity distinct from GR. Das et al. (2018) demonstrated that RG is equal to Einstein gravity and offered some cosmological implications in the structure of modified RG. Recently, Golovnev (2024) claimed that Rastall gravity may be equivalent to General Relativity by analyzing the claims of Visser (2018) and Darabi et al. (2018). The derivation of the action for Rastall gravity is subjected to the condition $\sqrt{-g} = 1$ (Golovnev, 2024). Holographic dark energy in RG was introduced by Ghaffari et al. (2020), considering vacuum energy, which serves as DE. They accepted the idea that the current accelerating cosmos is caused by the sum of this energy and the Rastall term. Assuming that the universe comprises interacting/non-interacting dark energy (DE) and dark matter (DM), Saleem & Shahnila (2021) investigated the phenomenon of cosmic evolution using curved FLRW spacetime bounded by an apparent horizon with a particular holographic cut-off in the framework of Rastall gravity. Saleem et al. (2024) recently introduced constant-roll warm inflation, a novel method for determining the precise inflationary solutions to the Friedman equations within the framework of Rastall theory of gravity (RTG). Using a linear parameterization of the Equation of State (EoS) in the FLRW background, Singh et al. (2024) examined the evolution of the universe within the framework of RTG. In the framework of RTG, Saleem & Hassan (2020) studied the dynamics of warm inflation induced by vector fields and concluded that the modified theory was compatible with the 2018 Planck observational data. In various contexts, several researchers have examined cosmological scenarios within the framework of Rastall gravity (Banerjee et al., 2023; Tangphati et al., 2023, 2024; Banerjee et al., 2024; Shaily et al., 2024; Singh et al., 2021, 2022; Singh & Pradhan, 2023; Singh & Mishra, 2020; Akarsu et al., 2020). This theory has received widespread attention in discussions and explanations of different phenomena related to the evolution of the universe. We aim to study the evolution of the universe exhibiting oscillatory expansion by analyzing its thermodynamic behavior. It is worth mentioning that Rastall gravity possesses oscillatory solutions subjected to a fixed point, which are marginally stable (Singh et al., 2021, 2022).

Models of dark energy that permit the equation of state parameter, w, to cross the "phantom divide line" (w=-1) investigate possibilities in which the expansion of the universe can go from accelerating to accelerating at an ever-increasing rate, which could result in a "Big Rip" (Astashenok & Tepliakov, 2022). The long-term destiny of the universe depends on our ability to comprehend dark energy and its behavior, particularly the potential for crossing the phantom divided line (PDL). Mohseni Sadjadi (2009) discusses the conditions required to cross the phantom divide line in a closed Friedmann-Robertson-Walker universe using an interacting holographic dark energy model. Huang et al. (2010) reconstitute the dark energy parameters from the most recent 397 Sne Ia, CMB, and BAO using three distinct parameterized dark energy models with the specified current matter density, Ω_{m0}. They discovered that an evolving dark energy with a crossing of the phantom dividing line is preferred when Ω_{m0} is not modest, such as when $\Omega_{m0} = 0.28$ or 0.32. For a more in-depth understanding of this discovery, we recommend the following papers Fang et al. (2008); Wei & Cai (2006); Ahmed & Pradhan (2020); Varshney et al. (2021); Dixit et al. (2022); Debnath et al. (2022).

The acceleration of the expansion of the universe, which defies the predictions of General Relativity (GR) without including dark energy, is the driving force behind this investigation. Consequently, modified gravity theories have been examined, such as Rastall gravity, which adds a non-minimal link between geometry and matter and a non-conserved stress-energy tensor. The goal of the authors' cyclic universe model is to address problems such as the initial singularity and future Big Rip scenarios, in addition to Quintom behavior, which is the transition between quintessence and phantom phases with a crossing of the phantom dividing line.

The oscillating Hubble parameter that alternates between positive and negative values across cycles is the result of the authors' proposed periodic parametrization of the deceleration parameter in the current model. With an anticipated occurrence at approximately 8.7 Gyr, this periodic behavior effectively simulates cosmic transit by suggesting that the cosmos experiences periods of acceleration and deceleration. In addition to the area around the initial singularity and future Big Rip, the model satisfies causality. To ensure that the model is physically feasible, thermodynamic characteristics such as entropy and energy conditions were examined.

Owing to the ad hoc periodic parameterization of the deceleration parameter q with the correct sign flipping as $q = m\cos kt - 1$ (Shen & Zhao, 2014), the universe accelerates after an epoch of deceleration (for every single cycle), which agrees with recent observations. Such periodic form of q leads to the following forms for the Hubble parameter and scale factor

$$H = \frac{k}{m\sin kt + c}\ , \quad a = a_0\left[\tan(\frac{1}{2}kt)\right]^{\frac{1}{m}}, \tag{2}$$

where $m > 0$, $k > 0$ and k acts as a cosmic frequency parameter. This specific form of the deceleration parameter was first introduced by Shen & Zhao (2014), who constructed a new oscillating Quintom Model. Using the well known redshift relation $z = \frac{1}{a} - 1$ we get

$$t = \frac{2}{k}\tan^{-1}\frac{1}{a_0^m(z+1)^m}. \tag{3}$$

Observations suggest that the signature change of q occurs at $z = 0.64$ for $m = 1.55$ (Capozziello et al., 2014; Capozziello et al., 2015; Farooq et al., 2017; Moraes et al., 2016). Since cosmic transit happens when $q = 0$ ($i.e.\ \ddot{a} = 0$), we have

$$t_{q=0} = \frac{1}{k}\cos^{-1}\frac{1}{m} \approx 8.7\,\text{Gyr for } m = 1.55,\ k = 0.1. \tag{4}$$

The scale factor (2) suffers from a future Big Rip singularity, where the slope of $a(t)$ increases significantly and goes to infinity $\frac{da}{dt} \to \infty \Rightarrow$ which eventually tears apart the spacetime fabric.

2. Framework of the Model

Our goal is to obtain solutions by carefully considering the matter source and gravitational background. The basic idea behind Rastall gravity (Rastall, 1972) is that the energy-momentum conservation in GR can't be always valid in curved space-time and, instead of $T^{\mu\nu}_{;\mu} = 0$, we should have

$$T^{\mu\nu}_{;\mu} = \lambda R^{,\nu}, \tag{5}$$

which leads to the generalized field equations

$$G_{\mu\nu} + K\lambda g_{\mu\nu}R = KT_{\mu\nu}. \tag{6}$$

The GR is recovered for $\lambda = 0$. $G_{\mu\nu}$ is the Einstein tensor and K is the coupling constant. The FRW metric given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right], \tag{7}$$

where $a(t)$ is the scale factor and κ is equal to 0 for a flat universe. Applying equation (6) to the metric (7) we obtain the cosmological equations as

$$3(1-4K\lambda)H^2 - 6K\lambda\dot{H} + 3(1-2K\lambda)\frac{\kappa}{a^2} = K\rho, \tag{8}$$

$$3(1-4K\lambda)H^2 + 2(1-3K\lambda)\dot{H} + (1-6K\lambda)\frac{\kappa}{a^2} = -Kp. \tag{9}$$

In the current work, we are interested only in the observationally supported flat case where $\kappa = 0$ (Tegmark et al., 2004; Bennett et al., 2003; Spergel et al., 2003b; Ahmed & Alamri, 2019a). The energy density ρ, cosmic pressure p and EoS parameter ω are written as

$$\rho = -\frac{1}{K}\left(12\lambda KH^2 + 6\lambda K\dot{H} - 3H^2\right), \tag{10}$$

$$p = \frac{1}{K}\left(12\lambda KH^2 + 6\lambda K\dot{H} - 3H^2 - 2\dot{H}\right), \tag{11}$$

where the dot denotes differentiation w.r.t time and the EoS parameter $\omega = p/\rho$. Equations (10) and (11) show that

$$p = -\rho - \frac{2}{K}\dot{H}, \tag{12}$$

where $\dot{H}$ expresses the rate of change of the Hubble parameter as

$$\dot{H} = -\frac{K}{2}(\rho + p) = -\frac{K}{2}(\gamma\rho). \tag{13}$$

We have considered the EoS $p = (\gamma - 1)\rho$ with $\frac{2}{3} \leq \gamma \leq 2$. As pointed out by Bamba et al. (2018), inserting this equation for $\dot{H}$ in Eq. (8) for the flat case results in.

$$H^2 = \frac{K\rho(3K\lambda\gamma - 1)}{3(4k\lambda - 1)}. \tag{14}$$

The physical behavior of the energy density is shown in Figure 1. The pressure changes from positive in a decelerating era to negative in an accelerating era dominated by dark energy with negative pressure. As a function of redshift, The EoS parameter $\omega(z)$ meets -1 at $z = 0$. The time evolution of the EoS parameter reveals quintum behavior as it passes the phantom divide line in the far future. The Hubble parameter H decreases during expansion as $\dot{H} < 0$, whereas it increases during contraction as $\dot{H} > 0$. The general dynamical behavior can be deduced from the negative values of q. In each cycle, the EoS parameter $\omega(t)$ lies between -2.25 and $\frac{1}{3}$, which agrees with the observations (Komatsu et al., 2011; Eisenstein et al., 2005). It starts from a positive value (radiation-like era), continues to decrease to zero (dust era $\omega = 0$), and finally passes to negative values. After reaching the DE-dominated era, it crosses the cosmological constant boundary to the phantom era, where $\omega < -1$ possesses a Quintom feature. The Quintom dynamics associated with the crossing of the phantom divide line leads to $\omega < -1$ today, which is also observationally supported (Zhao et al., 2007). A cosmological bouncing Quintom model was studied by Cai et al. (2007). ω is an increasing function in the second half of the cosmic cycle, starting from -2.25. Some observations favor DE with ω less than -1 (Vikman, 2005; Allen et al., 2004; Riess et al., 2004). Whether DE can evolve into the phantom era has been extensively investigated. In Vikman (2005), it was shown that in a DE dominated universe, the transition of DE from $\omega \geq -1$ to $\omega < -1$ cannot be explained by models of classical scalar field dynamics unless more complicated physics is included (Armendáriz-Picón, 2004; Feng et al., 2005; Sahni & Shtanov, 2003).

3. Work Density and Entropy

In this section, we calculate and analyze some thermodynamic quantities. The $(n+1)$-dimensional FRW metric is written as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2_{n-1}\right), \tag{15}$$

where the line element of an $(n-1)$-dimensional unit sphere is denoted by Ω^2_{n-1}. This metric can also be written as (Bak & Rey, 2000)

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2\Omega^2_{n-1}, \tag{16}$$

where $\tilde{r} = a(t)r$, $x^0 = t$, $x^1 = r$ and the 2-dimensional metric $h_{ab} = diag(-1, \frac{a^2}{1-\kappa r^2})$. The apparent horizon is defined geometrically as an imaginary surface beyond which the null geodesic congruences recede from the observer (Saha, 2018). It is a dynamical structure that evolves with time and is determined by the relation $h_{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$. This leads to the expression of the apparent horizon's radius in terms of the Hubble parameter as

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{\kappa}{a^2}}}. \tag{17}$$

For a flat universe (the case considered here), the apparent horizon radius is the inverse of the Hubble parameter $1/H$, which is the same definition as the Hubble horizon $\tilde{r}_H$. That means

$$\tilde{r}_A = \tilde{r}_H = \frac{1}{H}, \quad \text{for} \quad \kappa = 0. \tag{18}$$

Consequently, the rate of change of the horizon radius $\dot{\tilde{r}}_A = -\frac{\dot{H}}{H^2} = -\tilde{r}^2_A\dot{H}$ or can be written in general as

$$\dot{\tilde{r}}_A = -\tilde{r}^3_A H\left(\dot{H} - \frac{\kappa}{a^2}\right). \tag{19}$$

The work density W is defined as the work performed by the cosmic volume change owing to the change in the radius of the apparent horizon. For the current model we get

$$W = -\frac{1}{2}T^{ab}h_{ab} = \frac{1}{2}(\rho - p) \tag{20}$$

$$= -\frac{1}{K}\left(12\lambda KH^2 + 6\lambda K\dot{H} - 3H^2 - \dot{H}\right). \tag{21}$$

In the case of FRW spacetime with a perfect fluid, the surface gravity κ_{sg} is also defined as

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b\tilde{r}_A) = -\frac{1}{\tilde{r}_A}\left(1 - \frac{1}{2}\frac{\dot{\tilde{r}}_A}{H\tilde{r}_A}\right), \tag{22}$$

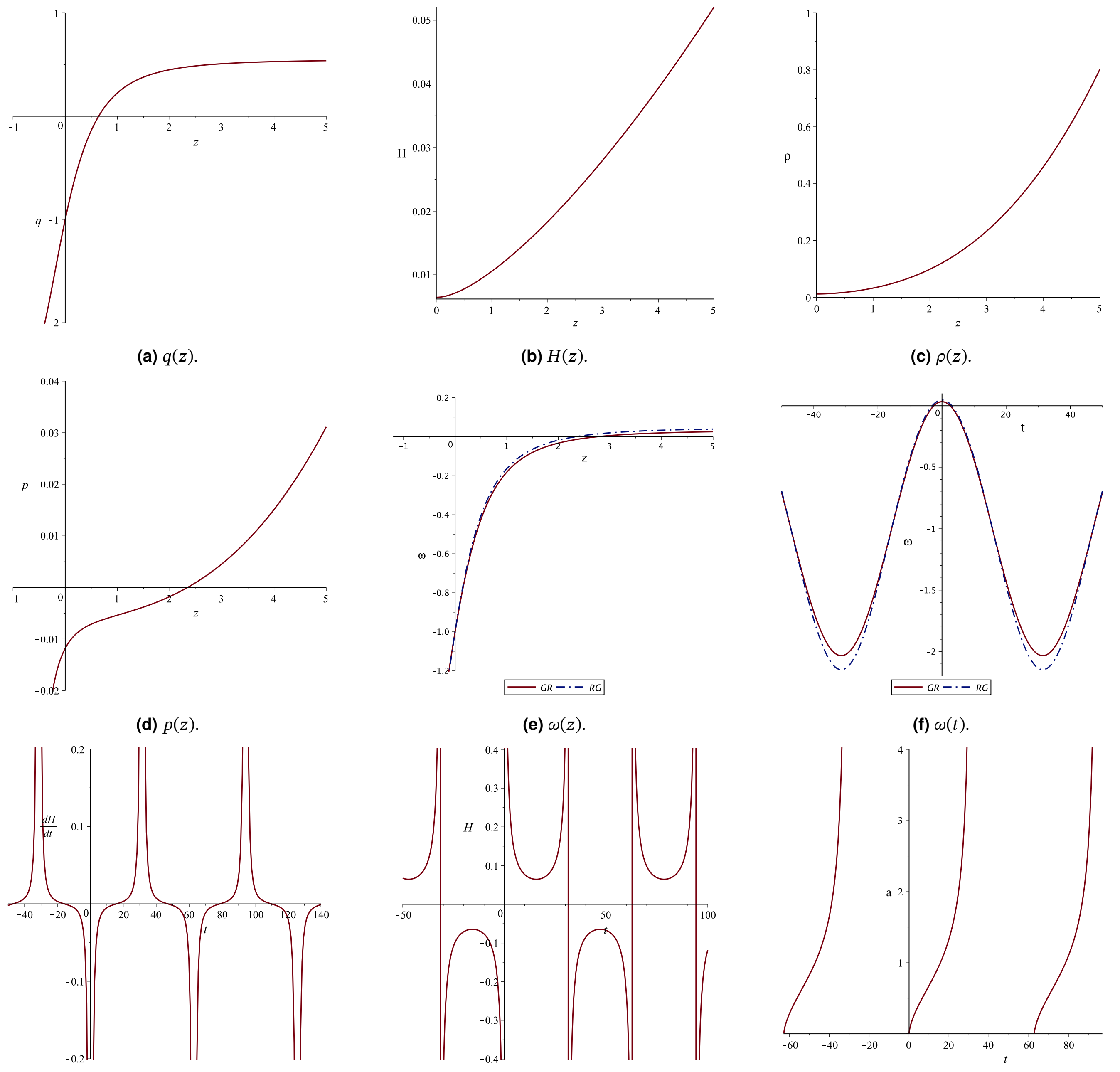
where h is the determinant of h_{ab}. In terms of κ_{sg}, the temperature on the horizon is given as $\frac{|\kappa_{sg}|}{2\pi}$

$$T_A = \frac{1}{2\pi\tilde{r}_A}\left|1 - \frac{1}{2}\frac{\dot{\tilde{r}}_A}{H\tilde{r}_A}\right|. \tag{23}$$

Therefore, the temperature depends on the Hubble parameter H, the radius of the apparent horizon $\tilde{r}_A$ and its rate of change during evolution $\dot{\tilde{r}}_A$ which is a result of the dynamical nature of the apparent horizon. If the change in the apparent horizon is very slow, then $\frac{\dot{\tilde{r}}_A}{H\tilde{r}_A} \ll 1$ and we obtain the expression for Hawking temperature $T_H = \frac{1}{2\pi\tilde{r}_A}$ which resembles the temperature of a spherically symmetric black hole with horizon radius $\tilde{r}_A$ (Sánchez & Quevedo, 2023).

The modified Bekenstein-Hawking entropy in Rastall gravity on the apparent horizon is given as (Bamba et al., 2018; Moradpour & Salako, 2016)

$$\tilde{S} = \left(1 + \frac{2\gamma}{4\gamma - 1}\right)S_0, \tag{24}$$



(a) $q(z)$. **(b)** $H(z)$. **(c)** $\rho(z)$. **(d)** $p(z)$. **(e)** $\omega(z)$. **(f)** $\omega(t)$. **(g)** $\dot{H}$. **(h)** $H(t)$. **(i)** $a(t)$.

Figure 1. Evolution of the model parameters against the redshift z (a) The DP $q(z)$ changes sign for $m > 1$ and equals -1 at $z = 0$. (b) The Hubble parameter $H(z)$. (c) Physical behavior of energy density $\rho(z)$. (d) A sign flipping in the evolution of cosmic pressure $p(z)$ (e) The EoS parameter $\omega(z)$ equals to -1 at $z = 0$. (f) $\omega(t)$ evolution shows a Quintum behavior, where it crosses the phantom divide line at very late times. GR stands for General Relativity and RG for Rastall Gravity. (g) The rate of change of the Hubble parameter $\dot{H} < 0$ during the expansion, which means a decreasing H, whereas this rate is > 0 during the contraction. (h) The Hubble parameter $H > 0$ during expansion and $H < 0$ during contraction. (i) The evolution of the scale factor shows an enormous increase over a short period of time (a future big rip). The unit of time t is taken as Gyr. Here $m = 1.55$, $\lambda = 1.4$, $K = 0.01$, $k = 0.1$ and $a_0 = 1$.

with $S_0 = \frac{A}{4}$ is the Bekenstein-Hawking entropy on the apparent horizon and $A = 4\pi\tilde{r}_A^2$ is the area. The normal Bekenstein-Hawking entropy in GR is recovered for $\gamma \to 0$. So, the area $A = 4\pi\tilde{r}_A^2$ is related to the modified area $\tilde{A}$ by $\tilde{A} = (1 + \frac{2\gamma}{4\gamma-1})A$ where the units $c = G = \hbar = 1$ has been considered. As a function of cosmic time t, the entropy should always be an increasing function, where the Universe evolves to the equilibrium state of maximum entropy. That means the two conditions

$$\dot{S} \geq 0 \quad \text{and} \quad \ddot{S} \leq 0, \tag{25}$$

should be satisfied. We recall that for a system at equilibrium, entropy has the maximum value, which means that it cannot increase anymore. On the other hand, any decrease is not possible as it violates the second law of thermodynamics which states that

$$d(S_m + S_h) > 0, \tag{26}$$

where S_h is the horizon entropy and S_m is the entropy of the entire matter field. At the apparent horizon, the FRW equations can be written as $dE = TdS + WdV$ where E is the total energy and $W = \frac{1}{2}(\rho - p)$ is the work density (Cai & Kim, 2005; Akbar & Cai, 2007). The cosmological work density is related to the cosmological constant in Ahmed & Alamri (2019b). The thermodynamic quantities w, $\tilde{r}_A$, $T_A =$ and S can be expressed as a function of the redshift z to probe their behavior. Figure 2 shows that the work density W has the same physical behavior as the energy density ρ versus z. In terms of cosmic time, it is a decreasing function during expansion, which means that the work done by the cosmic volume change decreases, resulting from variations in the apparent horizon radius. The first derivative of the event horizon radius is positive during expansion $\dot{\tilde{r}}_A > 0$ and negative during contraction $\dot{\tilde{r}}_A < 0$. A comparison between the evolution of the Hawking temperature T_H and the temperature on the horizon T_A as a function of z is shown in Figure 2c. Entropy $\tilde{S}$ is always positive. While the condition $\dot{\tilde{S}} > 0$ is satisfied, the non-positivity of the equilibrium condition $\ddot{\tilde{S}}$ exists only for the second half of the cosmic cycle.

4. Causality and Energy Conditions

One way to investigate physical acceptability is to test the classical linear energy conditions (ECs) (Hawking & Ellis, 1973; Wald, 1984), as the quantum corrections are ignored in the present model. The null, weak, strong, and dominant ECs are $\rho + p \geq 0$, $\rho \geq 0$, $\rho + p \geq 0$, $\rho + 3p \geq 0$ and $\rho \geq |p|$. These linear conditions cannot be satisfied in the presence of quantum effects (Martín-Moruno & Visser, 2013). The strong energy condition (SEC), for example, implies that gravity should always be attractive, which is not realistic when describing cosmic acceleration or inflation (Visser, 1997), (Visser, 1997). For the present model,

$$\begin{aligned}
\rho + p &= -\frac{2}{K}\dot{H}, \\
\rho - p &= -\frac{2}{K}(12\lambda K H^2 + 6\lambda K \dot{H} - 3H^2 - \dot{H}), \\
\rho + 3p &= \frac{6}{K}(4\lambda K H^2 + 2\lambda K \dot{H} - H^2 - \dot{H}).
\end{aligned}$$

The SEC is not expected to be valid during the negative pressure-dominated accelerating era, as negative pressure represents a repulsive gravity effect. The DEC is valid at all times for this model, which is expected, as it implies the non-negativity of the energy density. The SEC is valid only with the domination of attractive gravity in the decelerating era (at the first half of each cycle), and then becomes invalid with the domination of repulsive gravity during the accelerating era.

The causality condition for the adiabatic square sound speed $0 \leq \frac{dp}{d\rho} \leq 1$ should be satisfied through cosmic evolution. For the current model, using (10) and (11), we get

$$c_s^2 = -\frac{1}{3}\frac{12\lambda K H\dot{H} + 3\lambda K \ddot{H} - 3H\dot{H} - \ddot{H}}{4\lambda K H\dot{H} + \lambda K \ddot{H} - H\dot{H}}. \tag{27}$$

Figure 2h shows that the causality is satisfied except near the initial and the future Big Rip singularities

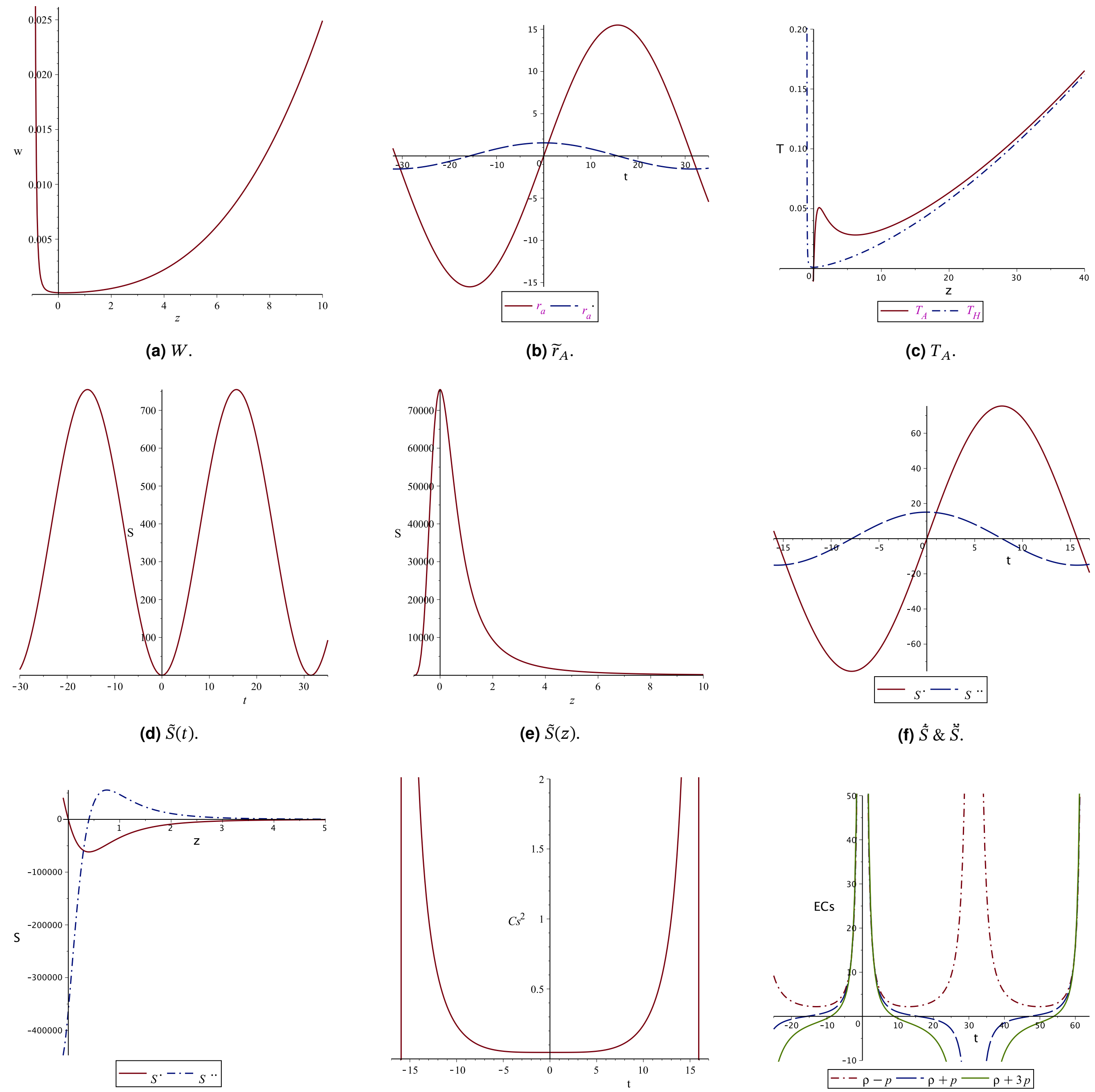
5. Conclusions

We presented a cyclic cosmological model of a flat universe with a future Big Rip singularity in the framework of Rastall gravity. The model possesses a Quintom behavior where the EoS parameter crosses the phantom divide line (cosmological constant boundary) at $\omega = -1$. In such oscillatory cosmic evolution, the deceleration-to-acceleration cosmic transit occurs at approximately 8.7Gyr. A recent analysis that included data from cosmic chronometers and the Pantheon compilation of supernovae calculated the transition redshift to be $z_t = 0.61^{+0.24}_{-0.16}$ (Yang & Gong, 2020). Given that the universe is currently 13.8 Gyr old, this correlates to a transition period of roughly 7.5 to 9.8 Gyr ago. A transition redshift of $z_t = 0.60^{+0.21}_{-0.12}$ was discovered by another study that looked at a parametrization of the deceleration parameter (Arora & Sahoo, 2024). This estimate, which places the transition era between 7.7 and 9.6 Gyr ago, is in good agreement with the previous one. A cosmic deceleration-to-acceleration transition that occurred approximately 8.7 Gyr ago is within the observed range based on these observational estimations. Thus, our conclusion seems to be in accordance with recent evidence from observations.

The EoS parameter ω meets -1 at redshift $z = 0$. The model exhibits the correct physical evolution of energy density and a sign flipping of cosmic pressure compatible with cosmic transit. Observational evidence supports that the EoS parameter ω is close to -1 at the present epoch (redshift $z = 0$) (Tripathi et al., 2017; Dinda & Banerjee, 2024).

The first derivative of the Hubble parameter $\dot{H}$ is negative during expansion, which means a decreasing H, and positive during contraction, which means an increasing H. The EoS parameter ω lies in the range $-2.25 \leq \omega(t) \lesssim \frac{1}{3}$. It begins in the radiation-like era, then passes through the dust era, the current dark energy-dominated era, before crossing the cosmological constant boundary to the phantom era. Simultaneously, plotting the ω evolution in Rastall Gravity (RG) and General relativity (GR) together shows a tiny difference between the two curves.

The evolution of the temperature on the dynamical horizon and Hawking temperature has been plotted against



(a) W. **(b)** $\widetilde{r}_A$. **(c)** T_A.

(d) $\tilde{S}(t)$. **(e)** $\tilde{S}(z)$. **(f)** $\dot{\tilde{S}}$ & $\ddot{\tilde{S}}$.

(g) $\dot{\tilde{S}}$ & $\ddot{\tilde{S}}$. **(h)** C_s^2. **(i)** ECs.

Figure 2. (a) Work density versus redshift. (b) Radius of the apparent horizon versus redshift. (c) Temperature on dynamical horizon versus Hawking temperature of a spherically symmetric black hole. (d) & (e) evolution of entropy versus t and z shows that $\tilde{S} \geq 0$. (f) $\dot{\tilde{S}}(t)$ and $\ddot{\tilde{S}}(t)$, while the condition $\dot{\tilde{S}} > 0$ is satisfied, the non-positivity of the equilibrium condition $\ddot{\tilde{S}} < 0$ exists only for the second half of cosmic cycle. (g) $\dot{\tilde{S}}(z)$ and $\ddot{\tilde{S}}(z)$. A violation of $\ddot{\tilde{S}}(t) < 0$ starts at $z \gtrapprox 0.4$ (h) The causality condition $0 \leq \frac{dp}{d\rho} \leq 1$ is satisfied except near the initial and the future Big Rip singularities (i) Linear energy conditions. Here $m = 1.55$, $\lambda = 1.4$, $K = 0.01$, $k = 0.1$ and $a_0 = 1$.

redshift. The entropy in the model was always positive. It starts as an increasing function of time in the first expanding half-cycle until it reaches the maximum value corresponding to the equilibrium state. The rate of change of entropy along with the equilibrium condition was calculated and plotted. The causality is satisfied all the time except near the initial singularity and the future Big Rip singularity.

In order to solve problems such as the initial singularity and future big rip situations, the oscillating Rastall universe model sets itself apart by including Rastall gravity to produce a cyclic cosmological model with

Quintom behavior. However, other studies have used alternative processes to help transcend the phantom divide, including parameterized post-Friedmann approaches, two-field models, holographic dark energy models, and gravity modifications such as $f(Q)$ gravity. To address the difficulties in simulating the accelerated expansion of the universe and the dynamic characteristics of the equation of state for dark energy, each framework provides distinct perspectives and approaches. The Introduction contains a comprehensive discussion with up-to-date references.

It is pertinent to investigate an oscillating Rastall universe that crosses the phantom division line because it provides a theoretical foundation for investigating dynamic dark energy models and possible gravitational alterations. Therefore, our research is useful and pertinent to the oscillating Rastall cosmos.

The authors (A. Pradhan & A. Dixit) are grateful for the resources and support provided during their visit to the Inter-University Centre for Astronomy & Astrophysics (IUCAA), Pune, India, as part of their Associateship program. To improve the manuscript in its current form, the authors would like to thank the anonymous reviewer for their insightful remarks.

References

Ahmed, N., & Alamri, S. Z. 2019a, IJGMMP, 16, 1950159, doi: 10.1142/S0219887819501597
—. 2019b, CJPh, 97, 1075, doi: 10.1139/cjp-2018-0635
Ahmed, N., Alfreedi, A. A., & Alzulaibani, A. A. 2023, IJGMMP, 20, 2350053, doi: 10.1142/S0219887823500536
Ahmed, N., Bamba, K., & Salama, F. 2020, IJGMMP, 17, 2050075, doi: 10.1142/S0219887820500759
Ahmed, N., Kamel, T. M., & Nouh, M. I. 2022, RMxAA, 58, 245, doi: 10.22201/ia.01851101p.2022.58.02.07
Ahmed, N., & Pradhan, A. 2020, NewA, 80, 101406, doi: 10.1016/j.newast.2020.101406
—. 2022, InJPh, 96, 301, doi: 10.1007/s12648-020-01948-4
Ahmed, N., & Rafat, H. 2018, IJGMMP, 15, 1850131, doi: 10.1142/S0219887818501311
Akarsu, Ö., Katırcı, N., Kumar, S., et al. 2020, EPJC, 80, 1050, doi: 10.1140/epjc/s10052-020-08586-4
Akbar, M., & Cai, R.-G. 2007, PhRvD, 75, 084003, doi: 10.1103/PhysRevD.75.084003
Allen, S. W., Schmidt, R. W., Ebeling, H., Fabian, A. C., & Van Speybroeck, L. 2004, MNRAS, 353, 457, doi: 10.1111/j.1365-2966.2004.08080.x
Armendáriz-Picón, C. 2004, JCAP, 2004, 007, doi: 10.1088/1475-7516/2004/07/007
Arora, S., & Sahoo, P. 2024, PDU, 45, 101510, doi: https://doi.org/10.1016/j.dark.2024.101510
Astashenok, A. V., & Tepliakov, A. 2022, Univ, 8, 265, doi: 10.3390/universe8050265
Astier, P., Guy, J., Regnault, N., et al. 2006, A&A, 447, 31, doi: 10.1051/0004-6361:20054185
Bak, D., & Rey, S.-J. 2000, CQGra, 17, L83, doi: 10.1088/0264-9381/17/15/101
Bamba, K., Jawad, A., Rafique, S., & Moradpour, H. 2018, EPJC, 78, 986, doi: 10.1140/epjc/s10052-018-6446-0
Bamba, K., Makarenko, A. N., Myagky, A. N., Nojiri, S., & Odintsov, S. D. 2014a, JCAP, 2014, 008, doi: 10.1088/1475-7516/2014/01/008
—. 2014b, JCAP, 2014, 008, doi: 10.1088/1475-7516/2014/01/008
Bamba, K., Makarenko, A. N., Myagky, A. N., & Odintsov, S. D. 2015, JCAP, 2015, 001, doi: 10.1088/1475-7516/2015/04/001
Banerjee, A., Islam, S., Dixit, A., & Pradhan, A. 2024, GrCo, 30, 496, doi: 10.1134/S0202289324700397
Banerjee, A., Tangphati, T., Hansraj, S., & Pradhan, A. 2023, AnPhy, 451, 169267, doi: https://doi.org/10.1016/j.aop.2023.169267
Bennett, C. L., Halpern, M., Hinshaw, G., et al. 2003, ApJS, 148, 1, doi: 10.1086/377253
Cai, R.-G., & Kim, S. P. 2005, JHEP, 2005, 050, doi: 10.1088/1126-6708/2005/02/050
Cai, Y.-F., Qiu, T., Zhang, X., Piao, Y.-S., & Li, M. 2007, JHEP, 2007, 071, doi: 10.1088/1126-6708/2007/10/071
Capozziello, S., Farooq, O., Luongo, O., & Ratra, B. 2014, PhRvD, 90, 044016, doi: 10.1103/PhysRevD.90.044016
Capozziello, S., Luongo, O., & Saridakis, E. N. 2015, PhRvD, 91, 124037, doi: 10.1103/PhysRevD.91.124037
Copeland, E. J., Sami, M., & Tsujikawa, S. 2006, IJMPD, 15, 1753, doi: 10.1142/S021827180600942X
Darabi, F., Moradpour, H., Licata, I., Heydarzade, Y., & Corda, C. 2018, EPJC, 78, 25, doi: 10.1140/epjc/s10052-017-5502-5
Das, D., Dutta, S., & Chakraborty, S. 2018, EPJC, 78, 810, doi: 10.1140/epjc/s10052-018-6293-z
de la Cruz-Dombriz, A., Farrugia, G., Said, J. L., & Gómez, D. S.-C. 2018, PhRvD, 97, 104040, doi: 10.1103/PhysRevD.97.104040
De Moraes, W. A. G., & Santos, A. F. 2019, GeReG, 51, 167, doi: 10.1007/s10714-019-2652-9
Debnath, U., Chakraborty, S., Maity, S., & Pradhan, A. 2022, IJMPA, 37, 2250198, doi: 10.1142/S0217751X22501986
Dinda, B. R., & Banerjee, N. 2024, EPJC, 84, 688, doi: 10.1140/epjc/s10052-024-13064-2
Dixit, A., Maurya, D. C., & Pradhan, A. 2022, IJGMMP, 19, 2250198, doi: 10.1142/S0219887822501985
Eisenstein, D. J., Zehavi, I., Hogg, D. W., et al. 2005, AJ, 633, 560, doi: 10.1086/466512
Fang, W., Hu, W., & Lewis, A. 2008, PhRvD, 78, 087303, doi: 10.1103/PhysRevD.78.087303
Farooq, O., Madiyar, F. R., Crandall, S., & Ratra, B. 2017, AJ, 835, 26, doi: 10.3847/1538-4357/835/1/26
Feng, B., Wang, X., & Zhang, X. 2005, PhLB, 607, 35, doi: https://doi.org/10.1016/j.physletb.2004.12.071
Ferraro, R., & Fiorini, F. 2007, PhRvD, 75, 084031, doi: 10.1103/PhysRevD.75.084031

Frampton, P. H. 2018, IJMPA, 33, 1844028, doi: 10.1142/S0217751X18440281

Ghaffari, S., Mamon, A. A., Moradpour, H., & Ziaie, A. H. 2020, MPLA, 35, 2050276, doi: 10.1142/S0217732320502764

Golovnev, A. 2024, AnPhy, 461, 169580, doi: https://doi.org/10.1016/j.aop.2023.169580

Hawking, S. W., & Ellis, G. F. R. 1973, The large-scale structure of space-time.

Hoyle, F., Burbidge, G., & Narlikar, J. V. 1993, ApJ, 410, 437, doi: 10.1086/172761

—. 1994, MNRAS, 267, 1007, doi: 10.1093/mnras/267.4.1007

—. 1995, RSPA, 448, 191, doi: 10.1098/rspa.1995.0012

Huang, T., Wu, P., & Yu, H. 2010, CPMA, 53, 419, doi: 10.1007/s11433-010-0125-y

Ijjas, A., & Steinhardt, P. J. 2018, CQGra, 35, 135004, doi: 10.1088/1361-6382/aac482

—. 2019, PhLB, 795, 666, doi: https://doi.org/10.1016/j.physletb.2019.06.056

Komatsu, E., Dunkley, J., Nolta, M. R., et al. 2009, ApJS, 180, 330, doi: 10.1088/0067-0049/180/2/330

Komatsu, E., Smith, K. M., Dunkley, J., et al. 2011, ApJS, 192, 18, doi: 10.1088/0067-0049/192/2/18

Majernik, V., & Richterek, L. 2006, arXiv e-prints, gr, doi: 10.48550/arXiv.gr-qc/0610070

Martín-Moruno, P., & Visser, M. 2013, JHEP, 2013, 50, doi: 10.1007/JHEP09(2013)050

Mohseni Sadjadi, H. 2009, EPJC, 62, 419, doi: 10.1140/epjc/s10052-009-1045-8

Moradpour, H., & Salako, I. G. 2016, AHEP, 2016, 3492796, doi: https://doi.org/10.1155/2016/3492796

Moraes, P. H. R. S., Ribeiro, G., & Correa, R. A. C. 2016, Ap&SS, 361, 227, doi: 10.1007/s10509-016-2788-y

Nojiri, S., & Odintsov, S. D. 2006, PhRvD, 74, 086005, doi: 10.1103/PhysRevD.74.086005

Nojiri, S., Odintsov, S. D., Oikonomou, V. K., & Paul, T. 2019, PhRvD, 100, 084056, doi: 10.1103/PhysRevD.100.084056

Nojiri, S., Odintsov, S. D., & Tretyakov, P. V. 2008, PThPS, 172, 81, doi: 10.1143/PTPS.172.81

Novello, M., & Bergliaffa, S. P. 2008, PhR, 463, 127, doi: 10.1016/j.physrep.2008.04.006

Overduin, J. M., & Wesson, P. S. 1997, PhR, 283, 303, doi: 10.1016/S0370-1573(96)00046-4

Pavlovic, P., & Sossich, M. 2017, PhRvD, 95, 103519, doi: 10.1103/PhysRevD.95.103519

Perlmutter, S., Aldering, G., Goldhaber, G., et al. 1999, AJ, 517, 565, doi: 10.1086/307221

Rastall, P. 1972, PhRvD, 6, 3357, doi: 10.1103/PhysRevD.6.3357

—. 1976, CJPh, 54, 66, doi: 10.1139/p76-008

Riess, A. G., Filippenko, A. V., Challis, P., et al. 1998, AJ, 116, 1009, doi: 10.1086/300499

Riess, A. G., Strolger, L.-G., Tonry, J., et al. 2004, AJ, 607, 665, doi: 10.1086/383612

Sachs, R., Narlikar, J. V., & Hoyle, F. 1996, A&A, 313, 703

Saha, S. 2018, Elements of Cosmological Thermodynamics, doi: 10.1007/978-3-319-74706-4

Sahni, V., & Shtanov, Y. 2003, JCAP, 2003, 014, doi: 10.1088/1475-7516/2003/11/014

Sahoo, P., Bhattacharjee, S., Tripathy, S. K., & Sahoo, P. K. 2020, MPLA, 35, 2050095, doi: 10.1142/S0217732320500959

Saleem, R., & Hassan, J. 2020, PDU, 28, 100515, doi: https://doi.org/10.1016/j.dark.2020.100515

Saleem, R., Shahid, I., Israr Aslam, M., & Wahab, A. 2024, NuPhB, 1009, 116728, doi: 10.1016/j.nuclphysb.2024.116728

Saleem, R., & Shahnila. 2021, PDU, 32, 100808, doi: https://doi.org/10.1016/j.dark.2021.100808

Shaily, Singh, A., Singh, J. K., Hussain, S., & Myrzakulov, R. 2024, Stability analysis of a dark energy model in Rastall gravity. https://arxiv.org/abs/2402.08709

Sharma, M. K., Pacif, S. K. J., Yergaliyeva, G., & Yesmakhanova, K. 2023, AnPhy, 454, 169345, doi: https://doi.org/10.1016/j.aop.2023.169345

Shen, M., & Zhao, L. 2014, ChPhL, 31, 010401, doi: 10.1088/0256-307X/31/1/010401

Singh, A., & Mishra, K. C. 2020, EPJP, 135, 752, doi: 10.1140/epjp/s13360-020-00783-0

Singh, A., & Pradhan, A. 2023, InJPh, 97, 631, doi: 10.1007/s12648-022-02406-z

Singh, A., Raushan, R., & Chaubey, R. 2021, CJPh, 99, 1073, doi: 10.1139/cjp-2020-0061

Singh, A., & Shukla, A. K. 2020, IJMPA, 35, 2050054, doi: 10.1142/S0217751X20500542

Singh, A., Singh, G. P., & Pradhan, A. 2022, IJMPA, 37, 2250104, doi: 10.1142/S0217751X22501044

Singh, J. K., Singh, A., , S., Ghosh, S. G., & Maharaj, S. D. 2024, PhyS, 99, 125001, doi: 10.1088/1402-4896/ad8841

Sotiriou, T. P. 2011, JPhCS, 283, 012034, doi: 10.1088/1742-6596/283/1/012034

Spergel, D. N., Verde, L., Peiris, H. V., et al. 2003a, ApJS, 148, 175, doi: 10.1086/377226

—. 2003b, ApJS, 148, 175, doi: 10.1086/377226

Steinhardt, P. J., & Turok, N. 2002, PhRvD, 65, 126003, doi: 10.1103/PhysRevD.65.126003

Sánchez, L. M., & Quevedo, H. 2023, PhLB, 839, 137778, doi: https://doi.org/10.1016/j.physletb.2023.137778

Tangphati, T., Banerjee, A., Hansraj, S., & Pradhan, A. 2023, AnPhy, 452, 169285, doi: https://doi.org/10.1016/j.aop.2023.169285

Tangphati, T., Banerjee, A., İzzet Sakallı, & Pradhan, A. 2024, ChJPh, 90, 422, doi: https://doi.org/10.1016/j.cjph.2024.03.045

Tegmark, M., Strauss, M. A., Blanton, M. R., et al. 2004, PhRvD, 69, 103501, doi: 10.1103/PhysRevD.69.103501

Tripathi, A., Sangwan, A., & Jassal, H. K. 2017, JCAP, 06, 012, doi: 10.1088/1475-7516/2017/06/012

Tripathy, S. K., Khuntia, R. K., & Parida, P. 2019, Bouncing cosmology in an Extended Theory of Gravity. https://arxiv.org/abs/1905.09477

Varshney, G., Sharma, U. K., Pradhan, A., & Kumar, N. 2021, ChJPh, 73, 56, doi: https://doi.org/10.1016/j.cjph.2021.04.014
Vikman, A. 2005, PhRvD, 71, 023515, doi: 10.1103/PhysRevD.71.023515
Visser, M. 1997, Sci, 276, 88, doi: 10.1126/science.276.5309.88
Visser, M. 1997, PhRvD, 56, 7578, doi: 10.1103/PhysRevD.56.7578
—. 2018, PhLB, 782, 83, doi: https://doi.org/10.1016/j.physletb.2018.05.028
Wald, R. M. 1984, General Relativity
Wei, H., & Cai, R.-G. 2006, PhLB, 634, 9, doi: https://doi.org/10.1016/j.physletb.2006.01.043
Yang, Y., & Gong, Y. 2020, JCAP, 2020, 059, doi: 10.1088/1475-7516/2020/06/059
Zhao, G.-B., Xia, J.-Q., Li, H., et al. 2007, PhLB, 648, 8, doi: https://doi.org/10.1016/j.physletb.2007.02.070