

USE OF APL IN RUNGE-KUTTA METHODS

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RESUMEN

En este trabajo se emplea APL para describir algoritmos de funciones clásicas Runge-Kutta, del cuarto al octavo orden, con y sin control del paso de integración, para resolver sistemas de ecuaciones diferenciales ordinarias del primer orden. Los resultados indican que para obtener gran precisión la fórmula Runge-Kutta del séptimo orden es la más eficiente. Para poca exactitud las fórmulas tradicionales, de cuarto orden, son más eficaces.

ABSTRACT

This paper uses APL to describe algorithms of classical Runge-Kutta functions up to the eighth-order with and without integration step size control for solving systems of ordinary differential equations of the first order. The results indicate that for high precision the best formula is the seventh-order Runge-Kutta. However, for low accuracy it is best to use the traditional Runge-Kutta formulas (fourth-order).

Key words: APL—DIFFERENTIAL EQUATIONS—RUNGE-KUTTA METHODS.

I. INTRODUCTION

where

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad (3)$$

and h is the integration step size.

For the formulas of Kutta (1901) we have:

$$\left. \begin{aligned} k_1 &= f(x_0, y_0)h \\ k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)h \\ k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)h \\ k_4 &= f(x_0 + h, y_0 + k_3), \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} k_1 &= f(x_0, y_0)h \\ k_2 &= f\left(x_0 + \frac{h}{3}, y_0 + \frac{k_1}{3}\right)h \\ k_3 &= f\left(x_0 + \frac{2h}{3}, y_0 - \frac{k_1}{3} + k_2\right)h \\ k_4 &= f(x_0 + h, y_0 + k_1 - k_2 + k_3), \end{aligned} \right\} \quad (4)$$

then

$$x_1 = x_0 + h, \quad y_1 = y_0 + \Delta y, \quad (2)$$

then

$$x_1 = x_0 + h, \quad y_1 = y_0 + \Delta y, \quad (5)$$

where

$$\Delta y = \frac{1}{8} (k_1 + 3k_2 + 3k_3 + k_4), \quad (6)$$

and again h is the integration step size.

The same formulas are used to compute y at x_2 , substituting x_1 and y_1 for x_0 and y_0 in either (1) or (4).

The extension from one equation to a system is very simple in APL (cf. Iverson 1962; Mendoza 1974). The following defined functions, R4 and K4, achieve this. Their left argument, N , is the number of differential equations. Their right argument, the vector Y , represents the initial value of the independent variable, the initial values of the dependent variables, the integration step size and the final value of the independent value, respectively.

The R4 function is:

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▽ N R4 Y;ΔY;K;K1;K2;K3;K4
[1] K1 ← Y[N+2]×DER Y[ιN+1]
[2] K2 ← Y[N+2]×DER Y[ιN+1]+
    (Y[N+2], K1) ÷ 2
[3] K3 ← Y[N+2]×DER Y[ιN+1]+
    (Y[N+2], K2) ÷ 2
[4] K4 ← Y[N+2]×DER Y[ιN+1]+
    Y[N+2], K3
[5] ΔY ← (+ + K ← (4, N) ρ K1,
    (2×K2, K3), K4) ÷ 6
[6] □ ← Y[ιN+1] ← Y[ιN+1]+
    Y[N+2], ΔY
[7] → Y[1]<Y[N+3]
▽,

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and the K4 function is:

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▽ N K4 Y;ΔY;K;K1;K2;K3;K4
[1] K1 ← Y[N+2]×DER Y[ιN+1]
[2] K2 ← Y[N+2]×DER Y[ιN+1]+
    (Y[N+2], K1) ÷ 3
[3] K3 ← Y[N+2]×DER (Y[1]+2×
    Y[N+2] ÷ 3), Y[1] ↓ ιN+1]+
    (-K1 ÷ 3) + K2
[4] K4 ← Y[N+2]×DER (Y[1]+Y[N+2]),
    Y[1] ↓ ιN+1]+ (K1-K2) + K3
[5] ΔY ← (+ + K ← (4, N) ρ K1,
    (3×K2, K3), K4) ÷ 8

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[6] □ ← Y[ιN+1] ← Y[ιN+1]+Y[N+2], ΔY
[7] → Y[1]<Y[N+3]
▽

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We use, as an example, R4 and K4 functions for solving the system

$$\left. \begin{aligned} \frac{dy}{dx} &= 2xy (\log_e z) \\ \frac{dz}{dx} &= 2xz (\log_e y) \end{aligned} \right\} \quad (7)$$

from $x = 0$ to $x = 5$ and $h = 0.1$

The system (7) has the exact solution

$$\left. \begin{aligned} y &= e^{\cos x^2} \\ z &= e^{\sin x^2} \end{aligned} \right\} \quad (8)$$

The function DER in R4 and K4 is defined as
 $\nabla Z \leftarrow DER Y$

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[1] Z ← 2×Y[1]×(Y[2]×⊗Y[3], -Y[3]×⊗Y[2])
▽,

```

and

$$Y \leftarrow 0 \ 2.7182818 \ 1 \ 0.1 \ 5. \quad (9)$$

A comparison of the computed and exact solution of system (7) is given in Table 1. The columns of this table list, first, the independent variable, x ; second and third, one computed dependent variable, y , obtained from R4 and K4, respectively; fourth and fifth, the other dependent variable, z , obtained from R4 and K4, respectively; from sixth to last, the computed minus the exact values of the dependent variables. The computed time in employing either R4 or K4 is, approximately, seven seconds on an IBM 370/155.

Table 1 shows clearly two known characteristics of fourth-order classical Runge-Kutta functions, namely, at the very beginning of a solution the error is approximately equal h^5 and when the solution is carried out further this error increases by accumulation. The latter can be prevented by either decreasing or controlling the integration step size.

TABLE 1

SOLUTION OF TWO DIFFERENTIAL EQUATIONS THROUGH R4 AND K4

x	y(R4)	y(K4)	z(R4)	z(K4)	$\Delta y(R4)$	$\Delta y(K4)$	$\Delta z(R4)$	$\Delta z(K4)$
0.1	2.7181460	2.7181458	1.0100500	1.0100500	0.0000001	-0.0000001	-0.0000000	0.0000000
0.2	2.7161089	2.7161078	1.0407997	1.0407997	0.0000005	-0.0000005	-0.0000000	-0.0000000
0.3	2.7073038	2.7073014	1.0940413	1.0940414	0.0000013	-0.0000010	-0.0000001	-0.0000000
0.4	2.6837855	2.6837815	1.1727107	1.1727109	0.0000028	-0.0000013	-0.0000004	-0.0000002
0.5	2.6350821	2.6350761	1.2806952	1.2806958	0.0000051	-0.0000009	-0.0000012	-0.0000006
0.6	2.5495072	2.5494992	1.4222957	1.4222972	0.0000083	0.0000003	-0.0000029	-0.0000014
0.7	2.4165426	2.4165328	1.6009899	1.6009933	0.0000120	0.0000023	-0.0000060	-0.0000026
0.8	2.2302250	2.2302142	1.8170050	1.8170120	0.0000150	0.0000042	-0.0000108	-0.0000037
0.9	1.9927313	1.9927200	2.0632438	2.0632571	0.0000155	0.0000042	-0.0000160	-0.0000027
1.0	1.7165385	1.7165265	2.3197587	2.3197807	0.0000128	0.0000008	-0.0000181	0.0000039
1.1	1.4233674	1.4233537	2.5487709	2.5488018	0.0000086	-0.0000050	-0.0000122	0.0000187
1.2	1.1393183	1.1393010	2.6951653	2.6951968	0.0000073	-0.0000100	0.0000032	0.0000347
1.3	0.8878890	0.8878663	2.6990781	2.6990881	0.0000115	-0.0000111	0.0000179	-0.0000279
1.4	0.6842544	0.6842257	2.5224125	2.5223730	0.0000180	-0.0000107	0.0000107	-0.0000288
1.5	0.5335838	0.5335498	2.1772371	2.1771456	0.0000184	-0.0000156	0.0000360	-0.0001275
1.6	0.4336260	0.4335892	1.7320227	1.7319230	0.0000069	-0.0000299	-0.0001134	-0.0002132
1.7	0.3796287	0.3795977	1.2824958	1.2824393	-0.0000169	-0.0000478	-0.0001780	-0.0002345
1.8	0.3696093	0.3696086	0.9062371	0.9062264	-0.0000543	-0.0000550	-0.0001865	-0.0001971
1.9	0.4095964	0.4096763	0.6365527	0.6365467	-0.0001206	-0.0000408	-0.0001415	-0.0001475
2.0	0.5198954	0.5201345	0.4690904	0.4690503	-0.0002517	-0.0000126	-0.0000738	-0.0001139
2.1	0.7419643	0.7424626	0.3849482	0.3848681	-0.0004843	0.0000140	-0.0000072	-0.0000873
2.2	1.1349376	1.1357758	0.3709298	0.3708413	-0.0007803	0.0000579	-0.0000469	-0.0000417
2.3	1.7254647	1.7265529	0.4327792	0.4327593	-0.0009107	0.0001775	0.0001046	0.0000848
2.4	2.3774329	2.3782729	0.6069181	0.6071488	-0.0005012	0.0003388	0.0001702	0.0004009
2.5	2.7173550	2.7171523	0.9673258	0.9682449	0.0005694	0.0003665	-0.0000394	0.0008797
2.6	2.4329035	2.4315080	1.5814181	1.5836461	0.0015178	0.0001223	-0.0009958	0.0012322
2.7	1.7079147	1.7059989	2.3271617	2.3304135	0.0012254	-0.0006905	-0.0011267	0.0021251
2.8	1.0151550	1.0130020	2.7188007	2.7206631	0.0010756	-0.0010774	0.0007846	0.0026469
2.9	0.5911249	0.5895474	2.3388605	2.3344206	0.0012286	-0.0003489	0.0007026	-0.0037373
3.0	0.4025788	0.4021347	1.5092506	1.5034610	0.0005092	0.0000652	-0.0007627	-0.0065523
3.1	0.3737384	0.3745464	0.8310779	0.8286596	-0.0004876	0.0003204	-0.0007208	-0.0031391
3.2	0.5018295	0.5051166	0.4829692	0.4819388	-0.0019046	0.0013826	0.0000364	-0.0009939
3.3	0.8953259	0.9024715	0.3704109	0.3700221	-0.0046581	0.0024875	0.0004775	0.0000886
3.4	1.7000194	1.7107296	0.4299434	0.4309904	-0.0073105	0.0033997	0.0003414	0.0013883
3.5	2.5825040	2.5878492	0.7321424	0.7369688	-0.0041648	0.0011805	-0.0004840	0.0043424
3.6	2.5234175	2.5138975	1.4610183	1.4749050	0.0052710	-0.0042490	-0.0064560	0.0074307
3.7	1.5471970	1.5347921	2.4552455	2.4715111	0.0062259	-0.0061789	-0.0084828	0.0077828
3.8	0.7484003	0.7385745	2.6017212	2.5955903	0.0062663	-0.0035594	0.0043615	-0.0017694
3.9	0.4191489	0.4173365	1.6144454	1.5837484	0.0037671	0.0019547	0.0021851	-0.0285120
4.0	0.3822846	0.3877940	0.7517815	0.7415797	-0.0015055	0.0040039	0.0019474	-0.0082544
4.1	0.6270388	0.6457753	0.4126430	0.4103215	-0.0094553	0.0092811	0.0028608	0.0005393
4.2	1.3999157	1.4344624	0.3940564	0.3978985	-0.0240345	0.0105122	0.0016508	0.0054928
4.3	2.5259873	2.5415329	0.6999360	0.7173111	-0.0239102	-0.0083646	-0.0034436	0.0139315
4.4	2.4059730	2.3623871	1.6009428	1.6476464	0.0130213	-0.0305647	-0.0290323	0.0176712
4.5	1.2020895	1.1678255	2.6580164	2.6768569	0.0173422	-0.0169218	-0.0212020	-0.0023615
4.6	0.5273025	0.5145984	2.1096966	2.0240858	0.0176508	0.0049467	0.0164714	-0.0691394
4.7	0.3712424	0.3828539	0.9190545	0.8818078	0.0015626	0.0131741	0.0130314	-0.0242153
4.8	0.5874951	0.6322667	0.4309543	0.4277973	-0.0199054	0.0248663	0.0106815	0.0075244
4.9	1.4764203	1.5502010	0.4098093	0.4261587	-0.0657850	0.0079957	0.0037643	0.0201138
5.0	2.6365790	2.6080100	0.8556226	0.9101903	-0.0578945	-0.0864635	-0.0204102	0.0341575

Section II will describe in APL (Mendoza 1974) classical Runge-Kutta formulas of the fifth—, sixth, seventh— and eighth— order which include an integration step size control. In section III a brief discussion is given.

II. RUNGE-KUTTA FORMULAS

a) Basic Equations

Let

$$Z \leftarrow F \quad X, Y \quad (10)$$

be an ordinary differential equation of the first order. We set

$$\left. \begin{aligned} F0 &\leftarrow F(X0, Y0) \\ F1 &\leftarrow F(X0+H \times A1, Y0+H \times B10 \times F0) \\ F2 &\leftarrow F(X0+H \times A2, Y0+H \times (B20 \times F0 \\ &\quad + B21 \times F1) \\ F3 &\leftarrow F(X0+H \times A3, Y0+H \times (B30 \times F0 \\ &\quad + (B31 \times F1) + B32 \times F2) \\ &\vdots \\ FK &\leftarrow F(X0+H \times AK, Y0+H \times (+BKJ \times FJ \\ &\quad (J \leftarrow K-1). \end{aligned} \right\} \quad (11)$$

We require

$$\left. \begin{aligned} Y &\leftarrow (Y0+H \times +/C \times FK) + \Delta \\ YN &\leftarrow (Y0+H \times +/D \times FK) + \Delta\Delta \end{aligned} \right\} \quad (12)$$

where H is the integration step size; K is 7, 9, 12 and 16 for the fifth—, sixth—, seventh— and eighth— order, respectively; Δ stands for those terms in a Taylor expansion with an H to the power 6, 7, 8 and 9; and $\Delta\Delta$, those terms with an H to the power 7, 8, 9 and 10, for the fifth—, sixth—, seventh— and eighth— order, respectively.

Equations (12) mean that we try to determine the coefficients A 's, B 's, C 's and D 's in such a way that

the first formula (12) represents the required order and the second expression of (12) the next order Runge-Kutta function.

We take, herein, the equations of condition for the Runge-Kutta coefficients given by Butcher (1964) and adopt the solution found by Fehlberg (1968).

b) Flow Chart

The flow chart is shown in Figure 1.

c) Description of the Flow Chart

- Box 1: Runge-Kutta-Fehlberg coefficients, quantities XX , H and the size of the matrix EFE are defined. The index AC is set equal to zero.
- Box 2: The index CRP is set equal to zero.
- Box 3: A decision is made to determine if $Y[N+1]$ is XX , then $Y[N+1]$ is printed, otherwise control is directed to Box 6.
- Box 4: The vector $Y[N+1]$ is printed and XX is assigned a new value.
- Box 5: A decision is made to determine if the independent variable has reached its final value and, accordingly, is directed to last or next box.
- Box 6: The printing control, quantity P , is defined.
- Box 7: The matrix EFE is computed and the index I is set equal to two. The quantities $YRES$, ERR and HH are defined. H receives a new value.
- Box 8: A decision is made according to the ERR value.
- Box 9: For ERR less than the tolerance, $Y[N+1]$ receives a new value and the accumulated error is increased by $ERR \times H$.
- Box 10: For ERR greater than the tolerance the index CRP is increased by one and control is directed to Box 7 or last.
- Box 11: The accumulated error is printed and calculation stops.

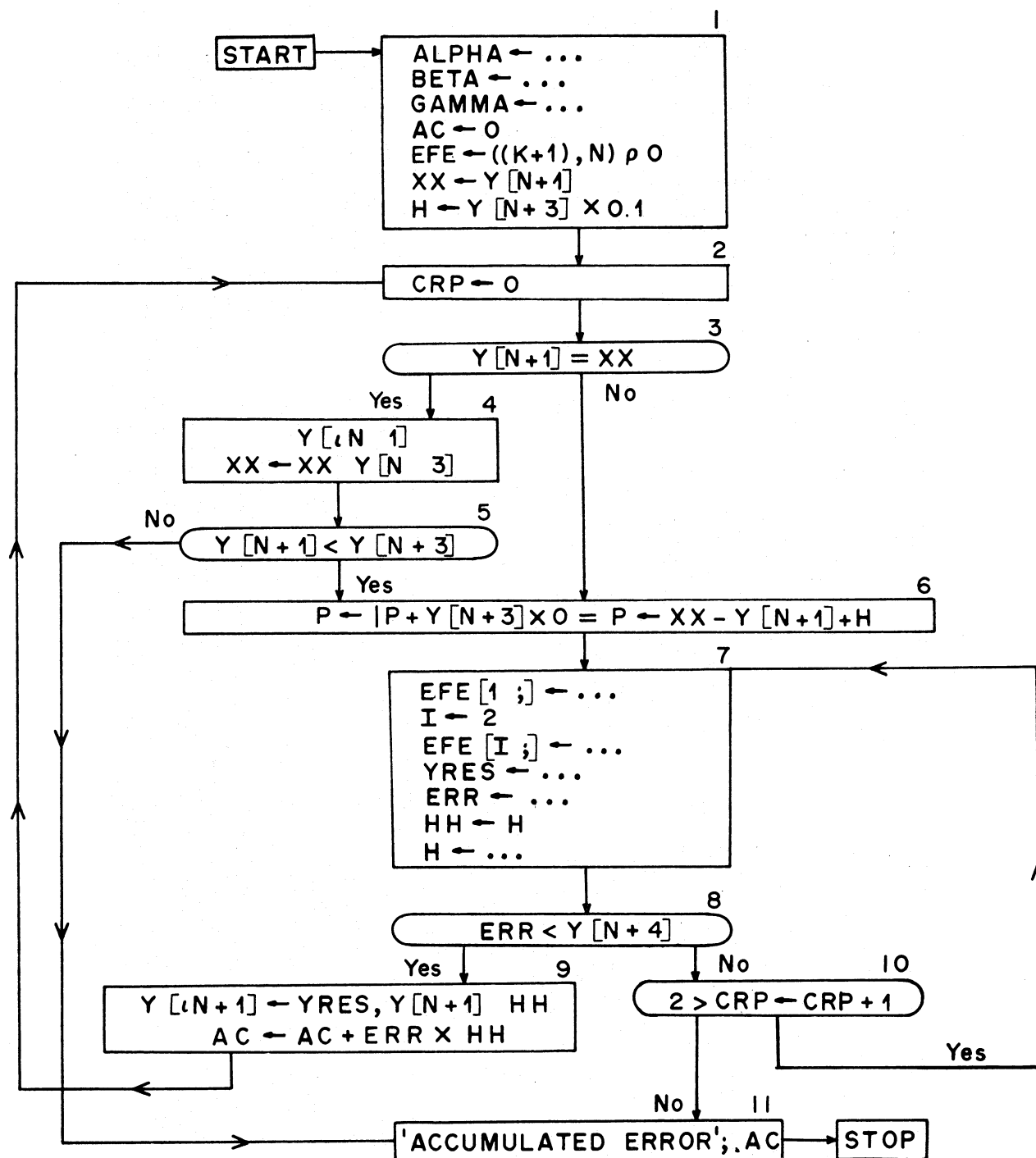


FIG. 1. The flow chart for Runge-Kutta functions.

d) *The fifth-order Runge-Kutta Function*

∇ N RK56 Y;AC;ALPHA;BETA;CRP;EFE;ERR;GAMMA;H;HH;I;P;XX;YRES

- [1] ALPHA $\leftarrow$ ($\div 6$), ($4 \div 15$), ($2 \div 3$), ($4 \div 5$), 1 0 1
- [2] BETA $\leftarrow$ ($\div 6$), ($4 \div 75$), ($16 \div 75$), ($5 \div 6$), ($8 \div 3$),
2.5, ($8 \div 5$), ($144 \div 25$), 4, ($16 \div 25$), ($361 \div 320$), ($18 \div 5$), ($407 \div 128$)
- [3] BETA $\leftarrow$ BETA, ($11 \div 80$), ($55 \div 128$), ($11 \div 640$), ($11 \div 256$), ($11 \div 160$),
($11 \div 256$), 0, ($93 \div 640$), ($18 \div 5$), $803 \div 256$
- [4] BETA $\leftarrow$ BETA, ($11 \div 160$), ($99 \div 256$), 0 1 0
- [5] GAMMA $\leftarrow$ ($31 \div 384$), 0, ($1125 \div 2816$), ($9 \div 32$), ($125 \div 768$), ($5 \div 66$), 0 0
- [6] AC $\leftarrow$ 0
- [7] EFE $\leftarrow$ (8, N) ρ 0
- [8] XX $\leftarrow$ Y[N+1]
- [9] H $\leftarrow$ Y[N+3] $\times$ 0.1
- [10] CRP $\leftarrow$ 0
- [11] $\rightarrow 15 - 3 \times Y[N+1] = XX$
- [12] Y[N+1]
- [13] XX $\leftarrow$ XX + Y[N+3]
- [14] $\rightarrow 28 - 13 \times Y[N+1] < Y[N+2]$
- [15] P $\leftarrow$ | P + Y[N+3] $\times$ 0 = P $\leftarrow$ XX - Y[N+1] + H
- [16] EFE[1:] $\leftarrow$ DER (N+1) ρ Y
- [17] I $\leftarrow$ 2
- [18] EFE[I:] $\leftarrow$ DER((N ρ Y) + ((I-1) ρ BETA[(I-1) + 0.5 $\times$ (I-2) $\times$ (I-1)] + . $\times$
EFE[I-1:] $\times$ H), Y[N+1] + ALPHA[I-1] $\times$ H
- [19] $\rightarrow 20 - 2 \times 8 \geq I \leftarrow I + 1$
- [20] YRES $\leftarrow$ (N ρ Y) + H $\times$ GAMMA + . $\times$ EFE
- [21] ERR $\leftarrow$ [/ | (EFE[1:] $\times$ ($31 \div 384$) - $7 \div 1408$) + (EFE[6:] - EFE[7:] + EFE[8:])
 $\times 5 \div 66$
- [22] HH $\leftarrow$ H
- [23] H $\leftarrow$ [/ P, HH $\times$ (Y[N+4] $\div$ (ERR $\times$ 20) + Y[N+4] $\times$ ($1E^{-16}$) $\star$ 0.2
- [24] $\rightarrow 27 - 2 \times ERR < Y[N+4]$
- [25] Y[N+1] $\leftarrow$ YRES, Y[N+1] + HH
- [26] $\rightarrow 10 + 0 \times AC \leftarrow AC + ERR \times HH$
- [27] $\rightarrow 28 - 13 \times 2 > CRP \leftarrow CRP + 1$
- [28] 'ACCUMULATED ERROR ' ; AC

∇

e) *The sixth-order Runge-Kutta Function*

∇ N RK67 Y;AC;ALPHA;BETA;CRP;EFE;ERR;GAMMA;H;HH;I;P;XX;YRES

- [1] ALPHA $\leftarrow$ ($2 \div 33$), ($4 \div 33$), ($2 \div 11$), 0.5, ($2 \div 3$), ($6 \div 7$), 1 0 1
- [2] BETA $\leftarrow$ ($2 \div 33$), 0, ($4 \div 33$), ($1 \div 22$), 0, ($3 \div 22$), ($43 \div 64$), 0,
($165 \div 64$), ($77 \div 32$), ($2383 \div 486$), 0, ($1067 \div 54$)
- [3] BETA $\leftarrow$ BETA, ($26312 \div 1701$), ($2176 \div 1701$), ($10077 \div 4802$), 0,
($5643 \div 686$), ($116259 \div 16807$), ($6240 \div 16807$), ($1053 \div 2401$)
- [4] BETA $\leftarrow$ BETA, ($733 \div 176$), 0, ($141 \div 8$), ($335763 \div 23296$), ($216 \div 77$),
($4617 \div 2816$), ($7203 \div 9152$), ($15 \div 352$)
- [5] BETA $\leftarrow$ BETA, 0 0, ($5445 \div 46592$), ($18 \div 77$), ($1215 \div 5632$),
($1029 \div 18304$), 0, ($1833 \div 352$), 0, ($141 \div 8$)
- [6] BETA $\leftarrow$ BETA, ($51237 \div 3584$), ($18 \div 7$), ($729 \div 512$), ($1029 \div 1408$), 0 1
- [7] GAMMA $\leftarrow$ ($77 \div 1440$), ($1771561 \div 6289920$), ($32 \div 105$), ($243 \div 2560$),
($16807 \div 74880$), ($11 \div 270$)
- [8] AC $\leftarrow$ 0
- [9] EFE $\leftarrow$ (10, N) ρ 0
- [10] XX $\leftarrow$ Y[N+1]
- [11] H $\leftarrow$ Y[N+3] $\times$ 0.1
- [12] CRP $\leftarrow$ 0
- [13] $\rightarrow 17 - 3 \times Y[N+1] = XX$
- [14] Y[N+1]
- [15] XX $\leftarrow$ XX + Y[N+3]
- [16] $\rightarrow 30 - 13 \times Y[N+1] < Y[N+2]$

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[17] P ← | P + Y[N+3] × 0 = P ← XX - Y[N+1] + H
[18] EFE[1;] ← DER(N+1)ρY
[19] I ← 2
    EFE[I-1;] × H), Y[N+1] + ALPHA[I-1] × H
[20] EFE[I;] ← DER((NρY) + ((I-1)ρBETA[(I-1) + 0.5 × (I-2) × I-1]) + . ×
    EFE[I-1;] × H), Y[N+1] + ALPHA[I-1] × H
[21] → 22 - 2 × 10 ≥ I ← I + 1
[22] YRES ← (NρY) + H × GAMMA + . × EFE[1,3+;5;]
[23] ERR ← [ / | (EFE[1;] × (77 ÷ 1440) - 11 ÷ 864) + (EFE[8;] - EFE[9;] + EFE[10;])
    × 11 ÷ 270
[24] HH ← H
[25] H ← [ / P, HH × (Y[N+4] ÷ (ERR × 20) + Y[N+4] × 1E-16) ★ ÷ 6
[26] → 29 - 2 × ERR < Y[N+4]
[27] Y[N+1] ← YRES, Y[N+1] + HH
[28] → 12 + 0 × AC ← AC + ERR × HH
[29] → 30 - 13 × 2 > CRP ← CRP + 1
[30] 'ACCUMULATED ERROR ' ; AC

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▽

f) The seventh-order Runge-Kutta Function

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▽ N RK78 Y;AC;ALPHA;BETA;CRP;EFE;ERR;GAMMA;H;HH;I;P;XX;YRES
[1] ALPHA ← (2 ÷ 27), (÷ 9), (÷ 6), (5 ÷ 12), 0.5, (5 ÷ 6),
    (÷ 6), (2 ÷ 3), (÷ 3), 1 0 1
[2] BETA ← (2 ÷ 27), (÷ 36), (÷ 12), (÷ 24), 0, (÷ 8),
    (5 ÷ 12), 0, (25 ÷ 16), (25 ÷ 16), (÷ 20), 0 0, 0.25 0.2, (25 ÷ 108), 0 0
[3] BETA ← BETA, (125 ÷ 108), (65 ÷ 27), (125 ÷ 54), (31 ÷ 300), 0 0 0, (61 ÷ 225),
    (2 ÷ 9), (13 ÷ 900), 2 0 0, (53 ÷ 6), (704 ÷ 45)
[4] BETA ← BETA, (107 ÷ 9), (67 ÷ 90), 3, (91 ÷ 108), 0 0, (23 ÷ 108), (976 ÷ 135),
    (311 ÷ 54), (19 ÷ 60), (17 ÷ 6), (- ÷ 12)
[5] BETA ← BETA, (2383 ÷ 4100), 0 0, (341 ÷ 164), (4496 ÷ 1025), (301 ÷ 82),
    (2133 ÷ 4100), (45 ÷ 82), (45 ÷ 164), (18 ÷ 41), (3 ÷ 205)
[6] BETA ← BETA, 0 0 0 0, (6 ÷ 41), (3 ÷ 205), (3 ÷ 41), (3 ÷ 41), (6 ÷ 41), 0,
    (1777 ÷ 4100), 0 0, (341 ÷ 164), (4496 ÷ 1025)
[7] BETA ← BETA, (289 ÷ 82), (2193 ÷ 4100), (51 ÷ 82), (33 ÷ 164), (12 ÷ 41), 0 1
[8] GAMMA ← (41 ÷ 840), (34 ÷ 105), (9 ÷ 35), (9 ÷ 35), (9 ÷ 280), (41 ÷ 840)
[9] AC ← 0
[10] EFE ← (13, N)ρ0
[11] XX ← Y[N+1]
[12] H ← Y[N+3] × 0.1
[13] CRP ← 0
[14] → 18 - 3 × Y[N+1] = XX
[15] Y[N+1]
[16] XX ← XX + Y[N+3]
[17] → 31 - 13 × Y[N+1] < Y[N+2]
[18] P ← | P + Y[N+3] × 0 = P ← XX - Y[N+1] + H
[19] EFE[1;] ← DER(N+1)ρY
[20] I ← 2
    EFE[I;] ← DER((NρY) + ((I-1)ρBETA[(I-1) + 0.5 × (I-2) × I-1]) + . ×
[21] → 23 - 2 × 13 ≥ I ← I + 1
[22] YRES ← (NρY) + H × GAMMA + . × EFE[1,5+;6;]
[23] ERR ← (41 ÷ 840) × [ / | EFE[1;] + EFE[11;] - EFE[12;] + EFE[13;]
[24] HH ← H
[25] H ← [ / P, HH × (Y[N+4] ÷ (ERR × 10) + 1E-7 × Y[N+4]) ★ ÷ 7
[26] → 30 - 2 × ERR < Y[N+4]
[27] Y[N+1] ← YRES, Y[N+1] + HH
[28] → 13 + 0 × AC ← AC + ERR × HH
[29] → 31 - 12 × 2 ≥ CRP ← CRP + 1
[30] EFE[I-1;] × H), Y[N+1] + ALPHA[I-1] × H
[31] 'ACCUMULATED ERROR ' ; AC

```

▽

g) *The eighth order Runge-Kutta Function*

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V N RK89 Y;AC;ALPHA;BETA;CRP;EFE;FRR;GAMMA;H;HH;I;P;XX;YRES

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[1] ALPHA+ 0.4436894037649818 0.6655341056474727 0.9983011584712091 0.3155 0.5054410094816907
    0.1714285714285714
[2] ALPHA+ALPHA, 0.8285714285714286 0.6654396612101156 0.2487831796806265 0.109 0.891 0.3995 0.6005
    1 0 1
[3] BETA+ 0.4436894037649818 0.1663835264118682 0.4991505792356046 0.2495752896178023 0 0.7487258688534068
[4] BETA+BETA, 0.206618911634006 0 0.1770788037798635 0.06819771541386949 0.1092782315266641
    0
[5] BETA+BETA, 0 0.004021596264236799 0.3921411816907898 0.09889928140916467 0 0 0.003513837022796397
[6] BETA+BETA, 0.1247609998316002 0.0557455468349898 0.368068652862422 0 0 0
[7] BETA+BETA, 2.227389746947601 1.374290825670291 2.04973900271116 0.4546796264134715
[8] BETA+BETA, 0 0 0 0.3254213170158915 0.2847666013852791 0.009783780167597915
[9] BETA+BETA, 0.06084207106262206 0 0 0 0 0.02118456574403701 0.1959655726617083
[10] BETA+BETA, 0.00427426403648176 0.01743436573681491 0.05405978329693192
[11] BETA+BETA, 0 0 0 0 0.1102982559782893 0.001256500852007256 0.003679004347758146 0.05778054277097207
[12] BETA+BETA, 0.1273247706866711 0 0 0 0 0.1144880500639611 0.2877302070969799 0.5094537945961136
[13] BETA+BETA, 0.1479968224437258 0.003652679387661674 0 0 0 0.08162989601231892 0.3860773563569351
[14] BETA+BETA, 0.03086224292460511 0.0580772545283206 0.3359865932888497 0.4106688040194996 0.01184024597235599
[15] BETA+BETA, 1.237535792124514 0 0 0 0 24.43076855135479 0.5477956893277866 4.441386353341325
[16] BETA+BETA, 10.01310481371327 14.99577310205176 5.894694852321701 1.738037750342898 27.51233069316673
[17] BETA+BETA, 0.3526085938833452 0 0 0 0 0.1839610314484827 0.6557018944974165 0.3908614488043986
[18] BETA+BETA, 0.2679464671285002 1.038302299138249 1.667232732425867 0.4955192585531598 1.139400113239706
[19] BETA+BETA, 0.05133669642465861 0.001046484734061481 0 0 0 0 0 0 0.006716388684499028
[20] BETA+BETA, 0.008182876218942502 0.004264034286448335
[21] BETA+BETA, 0.0002800902947416894 0.008783533387623868 0.01025450511082556 0 1.353655078617407
[22] BETA+BETA, 0 0 0 0 0.1839610314484827 0.6557018944974164 0.3908614488043986 0.2746628558129993
[23] BETA+BETA, 1.046485175357192 1.671496766712316 0.4952391682584181 1.14818364662733
[24] BETA+BETA, 0.04108219131383305 0 1
[25] GAMMA+ 0.03225608350021625 0.259837252837154 0.09284780599657703 0.1645233951476434
[26] GAMMA+GAMMA, 0.1766595163786007 0.2392010232035276 0.003948427460420285 0.03072649547586064
[27] AC+0
[28] EFE+(17,N)P0
[29] XX+Y[N+1]
[30] H+Y[N+3]*0.1
[31] CRP+0
[32] +36-3*Y[N+1]=XX
[33] Y[N+1]
[34] XX+XX+Y[N+3]
[35] +49-13*Y[N+1]<Y[N+2]
[36] P+|P+Y[N+3]*0=P+XX-Y[N+1]+H
[37] EFE[1;]+DER(N+1)P Y
[38] I+2
[39] EFE[I;]+DER({NoY}+(((I-1)PBETA[(I-1)+0.5*(I-2)*I-1]))+*EFE[(I-1;)]*H),Y[N+1]+ALPHA[I-1]*H
[40] +41-2*17>I+I+1
[41] YRES+(NoY)+H*GAMMA+*EFE[1,8+17;]
[42] FRR+[/]/(EFE[1 15 ;],-EFE[16 17 ;])*0.03072649547586064
[43] HH+H
[44] H+L/P,HH*(Y[N+4]+(ERR*10)+1E-8*Y[N+4])*0.125
[45] +48-2*ERR<Y[N+4]
[46] Y[N+1]+YRES,Y[N+1]+HH
[47] +31+0*AC+AC+ERR*HH
[48] +49-12*2>CRP+CRP+1
[49] 'ACCUMULATED ERROR ' ;AC

```


TABLE 2
COMPUTING TIME OF RUNGE-KUTTA
FUNCTIONS

n	R4		RK56		RK67		RK78	
	min	sec	min	sec	min	sec	min	sec
2	0	13	0	26	0	23	0	24
3	0	16	0	38	0	29	0	28
4	0	29	0	48	0	43	0	34
5	1	07	1	03	0	54	0	39
6	2	09	1	19	1	04	0	43
7	2	40	2	18	1	31	0	59
8	5	13	3	15	2	16	1	16
9	11	06	5	13	3	20	1	47
10	20	28	8	44	5	10	2	38
11	105	10	13	35	7	14	3	24

III. FINAL WORDS

The several Runge-Kutta functions given above can be compared to find out their computing time. We have constructed Table 2 to show this comparison using system (9), DER now defined as

$\nabla Z \leftarrow DER Y$
 [1] $Z \leftarrow 2 \times Y[3] \times (Y[1] \times \oplus Y[2]),$
 $- Y[2] \times \oplus Y[1]$
 ∇

and Y as

1 2.718281828459045 0 5 1 1E10

The columns of Table 2 contain, first, the number, n, of accurate decimal figures; second through last, the computing time of R4, RK56, RK67 and RK78, respectively. This time corresponds to the actual time spent in the calculation by an IBM 370/155 computer.

The results presented in Table 2 indicate that for low precision (less than 10^5) R4 (or K4) is the fastest algorithm; RK56, RK67 and RK78 have, approximately, the same speed. However, for high precision ($10^{10} - 10^{12}$) the best algorithm of all is RK78, twice faster than RK67, four times faster than RK56, and thirty times faster than R4, approximately.

APL works, approximately, 16-significant figures. On the other hand, the Runge-Kutta-Fehlberg coefficients of RK89 are irrational numbers which should be defined with more than 16-figures (Fehlberg 1968). This is reflected in the behavior of RK89, which is much slower than R4, RK56, RK67 and RK78 for all the values used to construct Table 2.

It is interesting to point out that RK56, RK67 and RK78 require less steps than Fehlberg's (1968) fifth—, sixth— and seventh-order formulas by, approximately, a factor of 2.

A word of caution: we have defined the above Runge-Kutta functions, RK56, RK67 and RK78, to be used with high precision. They may cause trouble at low accuracy. In this case, the integration step size control, H, should be redefined.

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