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THE LIFETIME OF BINARY STARS

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(Presented by S. J. Aarseth)

RESUMEN

El concepto de tiempo característico de una población de estrellas binarias se aplica solamente a aquellas de poca energía de amarre, esto es, a binarias separadas. Se encontró que de entre varios posibles métodos para estimar el tiempo característico, el mejor es el que se basa en el cambio promedio de la energía de amarre debido a encuentros que no rompen el sistema.

ABSTRACT

The concept of the "lifetime" of a population of binaries is applicable only to those of low binding energy, i.e. wide pairs. Of several possible methods of estimating a lifetime, that based on the average change in binding energy due to non-disruptive encounters appears to be the best.

I. INTRODUCTION

The observed abundance of wide binary stars in the solar neighbourhood has frequently led to a consideration of their lifetimes, since they must occasionally be subject to encounters with other stars. The question of lifetimes is also relevant to the study of stellar perturbations on the orbits of comets.

The problem can be thought of in the following terms. Let f $d\varepsilon$ be the number-density of binaries with binding energy ε and components with masses m_1, m_2.† Then as a result of encounters with stars of mass m_3, f changes according to

$$\frac{\partial f}{\partial t} = [n_1 n_2 n_3 Q(\varepsilon)] - n_3 Q(\varepsilon, -\infty) f + + n_3 \int_0^\infty d\varepsilon' \{f(\varepsilon') Q(\varepsilon', \varepsilon - \varepsilon') - f(\varepsilon) Q(\varepsilon, \varepsilon' - \varepsilon) \quad (1)$$

(Heggie 1975*a*), where n_i is the number-density of single stars of mass m_i, the first term (in square brackets) on the right-hand side corresponds to formation of new binaries, the second corresponds to direct destruction, the first part of the integral to encounters leading from energy ε' to ε, and the second part to those in the opposite direction.

If the velocities of single stars and of the centres of mass of binaries have appropriate Maxwellian distributions, an equilibrium solution of (1), i.e. satisfying $\partial f/\partial t = 0$, is given by the Boltzmann distribution, which we denote by f_0 (Heggie 1975*b*). However, since $f \gg f_0$ for wide binaries in the solar neighbourhood (Ambartsumian 1937), the "formation" term on the right-hand side of (1) may be ignored. Alternatively, we can regard f as the excess above the Boltzmann distribution.

In the problem of lifetimes we are concerned with casses in which

$$f(\varepsilon) = \delta(\varepsilon - \varepsilon_0) \quad (2)$$

initially, i.e. all pairs have energy ε_0. There are two aspects to the problem:

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† In the cometary case $m_2 = 0$ and a slightly different formulation is needed.

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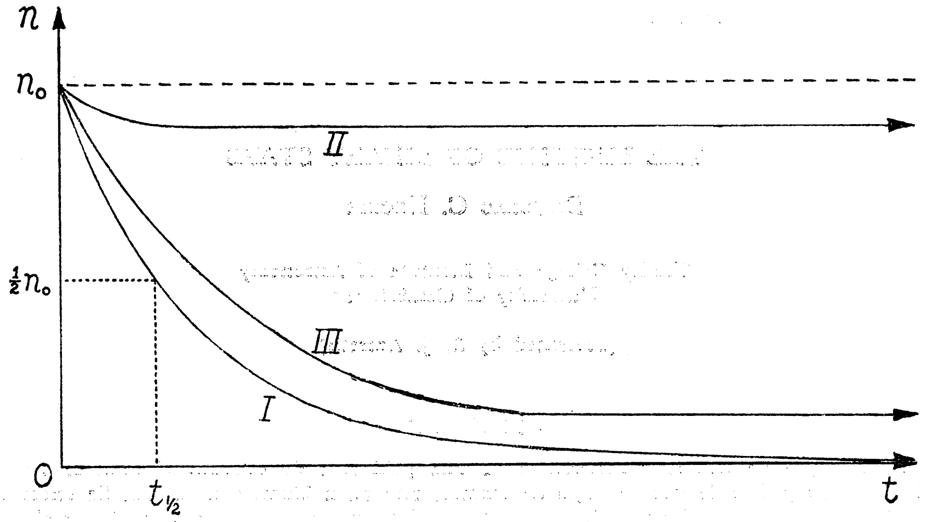

FIG. 1. Schematic dependence of the number-density, n, of binaries on time, t, where $n = n_0$ when $t = 0$. The three curves are described in the text, and $t_{\frac{1}{2}}$ is the half-life for curve I.

a) What is meant by "lifetime"?
b) How may it be calculated?

Neither of these is straightforward, and we shall consider them in turn.

II. THE MEANING OF LIFETIME

Let $n \equiv \int_0^\infty f(\varepsilon)\, d\varepsilon$ be the total number-density of all binaries. Its dependence on time *may* take a form decreasing asymptotically to zero, like curve I in Figure 1.

We might then define lifetime as a kind of "half-life", i.e. $t_{\frac{1}{2}}$ in the Figure, but this definition is of no use for very energetic (i.e. very "hard") binaries, which follow a curve like II: initially a few of these may be destroyed, but the remainder become still more energetic (Heggie 1975*a*) and survive indefinitely. Even for "soft" binaries the concept of "half-life" is of somewhat restricted significance, since a few of them may become very hard and thus no longer subject to disruption; they follow a curve like III.

An alternative which overcomes these difficulties is the definition

$$t_0 = -\left(\frac{1}{\frac{d \ln n}{dt}}\right)_{t=9}, \qquad (3)$$

derived from the initial slope of the curve of n against t. This definition of lifetime is based entirely on direct disruption from energy ε_0, and ignores the role of intermediate encounters which, cumulatively, *may* lead to disruption at a much higher rate. The following example illustrates this difficulty. Let

$$Q(\varepsilon, -\infty) = A/\varepsilon \qquad (4)$$

$$Q(\varepsilon, \varepsilon' - \varepsilon) = \begin{cases} B/\varepsilon^2 \text{ for } 0 \leqslant \varepsilon' < \varepsilon \\ 0 \text{ otherwise.} \end{cases} \qquad (5)$$

Though this choice might seem artificial, in fact (4) is the correct form for soft binaries (Heggie 1975*a*) and (5) has the correct form for $\varepsilon' \downarrow 0$, i.e., near the disruption barrier, where one might consider the role of intermediate encounters to be of special importance. Then it is shown in the Appendix that the solution of (1) and (2) yields

$$n = (1 - \alpha) \int_0^1 (1 - \xi)^{-\alpha} \exp(-p_0 n_3 t/\xi) \{(2 - \alpha)\xi + p_0 n_3 t\} d\xi, \qquad (6)$$

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where $\alpha \equiv B/(A+B)$, $p_0 \equiv (A+B)/\varepsilon_0$. Considering the limit $\alpha \downarrow 1$ at fixed t we find that $n \rightarrow (1 - p_0 n_3 t) \exp(p_0 n_3 t)$. Hence $t_{\frac{1}{2}} \sim \frac{1}{p_0 n_3}$ and so $t_{\frac{1}{2}} \ll t_0$ when $B \gg A$, i.e., t_0 is a very poor measure of lifetimes in this limit.

In conclusion it seems best to use $t_{\frac{1}{2}}$ as our measure of lifetime, but to restrict the concept to soft binaries only. Thus we remove the problem posed by curves like II in Figure 1. Equivalently, following Cruz-González and Poveda (1971), we may adopt $t_{1/e}$, the e-folding time for the number-density of binaries.

III. GENERAL CONSIDERATIONS IN THE CALCULATION OF LIFETIMES

Fresh difficulties arise when we come to calculate the lifetime theoretically. An exception is the quantity t_0, which is very easy to calculate, since $t_0 = 1/(n_3 Q(\varepsilon_0, -\infty))$, by (1). Although a formula of this type was used by Öpik (1973) and by Heggie (1975*a*), we have already seen that it *can* yield a severe overestimate of $t_{\frac{1}{2}}$.

The type of estimate which appears most frequently in the literature (e.g. Ambartsumian 1937) is of the form

$$t_1 = -\frac{\varepsilon_0}{n_3 \int_{-\infty}^{\infty} (\varepsilon - \varepsilon_0)\, Q(\varepsilon_0, \varepsilon - \varepsilon_0)\, d\varepsilon}, \quad (7)$$

which is based on the average rate at which energy is transferred to the binary by encounters. As Cruz-González and Poveda (1971) rightly pointed out, however, this formula may underestimate $t_{\frac{1}{2}}$ if encounters lead to a small number of very large, disruptive changes in energy, for in such cases the average binding energy of an evolved population of binaries could be close to zero even though almost all are still bound. For this reason one might consider a third possibility, namely

$$t_2 = -\frac{\varepsilon_0}{n_3 \int_{0}^{\infty} (\varepsilon - \varepsilon_0)\, Q(\varepsilon_0, \varepsilon - \varepsilon_0)\, d\varepsilon} \quad (8)$$

which differs from t_1 in that we only consider encounters which leave the binary still bound.

Note that we have offered little justification for the use of either t_1 or t_2 as an estimate of $t_{\frac{1}{2}}$, beyond the general understanding that the mean rate of change of the binding energy of a binary must be connected with its lifetime, in some sense. In principle it is necessary to solve (1) to find a formula for the lifetime, but even when we know approximate forms for the functions Q (Heggie 1975*a*), equation (1) is too difficult to solve. One potentially hopeful avenue for progress might be offered by a Fokker-Planck treatment of (1), the right-hand side being expanded in the form

$$n_3 \frac{\partial}{\partial \varepsilon}\left\{ f(\varepsilon) \int_{-\infty}^{\infty} Q(x, -y)\, y\, dy \right\} + \tfrac{1}{2} n_3 \frac{\partial^2}{\partial \varepsilon^2} \left\{ f(\varepsilon) \int_{-\infty}^{\infty} Q(x, -y)\, y^2 dy \right\} + \cdots$$

However, from the approximate expressions for Q already referred to, or else by methods similar to those of the following section in the present paper, it is easy to estimate that successive terms of this expansion are in the ratio

$$-(\beta\varepsilon)^{-1} \ln(\beta\varepsilon),\ (\beta\varepsilon)^{-2},\ (\beta\varepsilon)^{-3}, \ldots$$

where $\frac{3}{2}\beta^{-1}$ is the mean kinetic energy of the single stars, and we have estimated $\partial/\partial\varepsilon$ by $1/\varepsilon$. If $\beta\varepsilon \ll 1$, as is the case for very soft pairs, successive terms of this expansion increase rapidly, and neglect of the higher terms, which is necessary for the Fokker-Planck treatment, is not even justified in order of magnitude.

The reason for the failure of the direct Fokker-Planck treatment is related to that for preferring t_2 to t_1, and this offers some hope that a Fokker-Planck treatment restricted to non-disruptive encounters only may be justifiable. However, until this is demonstrated, if it can be, it seems unwarranted to supplement t_1 or t_2 by the effects of "diffusion", corresponding to the second term in the Fokker-Planck expansion of (1).

IV. CALCULATION OF LIFETIMES TO ORDER OF MAGNITUDE

Expressions for the three estimates of lifetime may be obtained from results of calculations of the

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functions Q (Heggie 1975*a*). However, since the latter results are only asymptotic expressions in certain limits, the estimates of t_1 and t_2 so obtained will be correct in order of magnitude only, certainly as far as non-dominant terms are concerned. If one is interested only in the integrals which appear in the denominators in the definitions of t_1 and t_2 then more refined results may be obtained. In this paper we shall confine ourselves to showing how the lifetimes may be estimated to order of magnitude.

Consider a binary with components of mass m_1, m_2, such that its initial semi-major axis is a, its initial binding energy is ε, and the initial relative velocity of its components is v. Let a third body of mass m_3 approach with velocity V relative to the centre of mass of the binary, in such a way that its impact parameters to the components and to the centre of mass of the binary are p_1, p_2 and p, respectively. Since we consider only very soft binaries we have $v \ll V$, and if we also assume that

$$\frac{v}{a} \ll \frac{V}{p + a} \tag{9}$$

then the encounter may be treated as impulsive.

First we consider the case in which

$$\frac{G(m_1 + m_3)}{V^2} \ll p_1 \ll a. \tag{10}$$

Then the motion of the third body is approximately rectilinear, and we readily estimate the impulsive change in v to be

$$\Delta v \sim \frac{Gm_3}{p_1^2}\frac{p_1}{V}. \tag{11}$$

The binary will be frequently disrupted if $\Delta v \gtrsim v$, i.e., if

$$p_1 \lesssim \frac{Gm_3}{Vv}. \tag{12}$$

It is not difficult to see that the right-hand side here lies within the range for p_1 specified by (10). Using the estimate

$$v^2 \sim \frac{m_1 + m_3}{m_1 m_2}\varepsilon \tag{13}$$

we find that the destruction cross-section is

$$\Sigma \sim \frac{G^2 m_1 m_2 m_3^2}{V^2(m_1 + m_2)\varepsilon},$$

whence

$$t_0 \sim \frac{(m_1 + m_2)V\varepsilon}{n_3 G^2 m_2 m_1 m_3^2}.$$

Now the change in energy of the binary is

$$\Delta\varepsilon \simeq -\tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}((\Delta \underset{\sim}{v})^2 + 2\underset{\sim}{v}\cdot\Delta\underset{\sim}{v}). \tag{14}$$

On the impulse approximation $\Delta\underset{\sim}{v}$ is independent of $\underset{\sim}{v}$, and so the average of the second term on the right vanishes. Hence from (8) one obtains

$$\langle\Delta\varepsilon\rangle \sim -\int \frac{m_1 m_2}{m_1 + m_2}\frac{G^2 m_3^2}{p_1^2 V^2} p_1 dp_1.$$

The range of p_1 for which this expression is approximately valid is given by (10), and so

$$\langle\Delta\varepsilon\rangle \sim -\frac{m_1 m_2}{m_1 + m_2}\frac{G^2 m_3^2}{V^2}\ln\frac{aV^2}{G(m_1 + m_3)}. \tag{15}$$

If, however, we consider only encounters which do not disrupt the binary, i.e. when (12) is reversed, our estimate becomes

$$\langle\Delta\varepsilon\rangle \sim -\frac{m_1 m_2}{m_1 + m_2}\frac{G^2 m_3^2}{V^2}\ln\frac{a^{\frac{1}{2}}V(m_1 + m_2)^{\frac{1}{2}}}{G^{\frac{1}{2}}m_3}, \tag{16}$$

which is of the same order as (15) if the masses are not too different. This means that $\langle\Delta\varepsilon\rangle$ for *all* close encounters is approximately twice that for non-disruptive close encounters only. Note, however, that in both cases there is another contribution from encounters close to the second component.

Now we consider the remaining more distant encounters with $a \ll p \ll Va/v$, the upper limit being inserted for consistency with (9). Since forces are now tidal we have $\Delta v \sim Gm_3 a/(p^2 V)$ and so

$$< \Delta\,\varepsilon > \sim -\int \frac{m_1 m_2}{m_1 + m_2}\frac{G^2 m_3^2 a^2}{p^4 V^2}\,p\,dp \sim$$
$$\sim -\frac{m_1 m_2}{m_1 + m_2}\frac{G^2 m_3^2}{V^2}.$$

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Thus the contribution from close encounters dominates by a logarithmic factor, whence

$$t_1 \sim \frac{(m_1 + m_2) V\varepsilon}{G^2 m_1 m_2 m_3^2 \ln (mV^2/\varepsilon)\, n_3}$$

and $t_2 \sim 2t_1$ since (15) and (16) are comparable. Here we have ignored different mass-factors in the logarithms.

These rough estimates of t_0, t_1 and t_2 have most of the important properties of the more refined estimates already referred to, differing from these only in numerical factors or non-dominant terms. In particular we see for very soft binaries that t_1 is considerably less than t_0, and we had anticipated that these woull be, respectively, an underestimate and an overestimate. They straddle the value of t_2, which might be regarded as the most satisfactory means of estimating $t_{\frac{1}{2}}$. However, the differences between the formulae, being logarithmic, are not great in practical cases. A numerical evaluation of t_0 will be found in Heggie 1975*a*.

I am indebted to Dr. S. J. Aarseth for reading this paper at the conference, and to him and to Prof. King, Prof. Poveda and Mrs. C. Allen for their interesting comments.

APPENDIX

Solution of equation (1) for a model problem.

In this appendix we shall sketch the solution of (1), with the creation term omitted, using (4) and (5) for the functions Q. However, we shall not proceed beyond the calculation of n, the total number-density of all binaries.

Taking the Laplace transform of (1) with respect to the time-variable $\tau \equiv n_3 t$, we find that

$$\hat{f} = \frac{f_0 + F(\varepsilon)}{p + (A + B)/\varepsilon} \quad , \qquad \text{(A1)}$$

where f_0 is the initial form of f, a ^ denotes the Laplace transform, and $F(\varepsilon) \equiv B \int_\varepsilon^\infty d\varepsilon' \, \hat{f}(\varepsilon') \varepsilon'^{-2}$.

Turning (A1) into a differential equation for F, this may be solved if we take (2) for f_0, whence

$$F = \begin{cases} \dfrac{B}{p\varepsilon_0^2} \left(\dfrac{\varepsilon_0}{\varepsilon_0 + (A+B)/p} \right)^{1 + \frac{B}{A+B}} \times \\ \times \left(\dfrac{\varepsilon}{\varepsilon + (A+B)/p} \right)^{-\frac{B}{A+B}} & \varepsilon < \varepsilon_0 \\ 0. & \varepsilon > \varepsilon_0 \end{cases}$$

Integrating (A1) over ε we now find that

$$\hat{n} = \frac{1}{p + p_0} + \frac{\alpha p_0^2}{p^3} \left(\frac{p}{p + p_0} \right)^{1+\alpha} \int_0^{p/p_0} \left(\frac{\xi}{1 + \xi} \right)^{1-\alpha} d\xi.$$

Despite appearances there is no singularity at $p = 0$, and the contour for the inversion of $\hat{n}$ can be taken into $\mathrm{Re}(p) < 0$. By splitting up the ξ – integration into two parts:

$$\int_0^{p/p_0} = \int_0^{-1} + \int_{-1}^{p/p_0} ,$$

we find that $\hat{n}$ becomes the sum of two functions, one of which is regular in $\mathrm{Re}(p) < 0$ and contributes nothing to the inversión, while the other, denoted by $\hat{n}_2$, has a branch point at $p = -p_0$. We find that

$$\hat{n}_2 = - \frac{\pi\alpha(1 - \alpha) p_0}{p^3} \left(\frac{-p}{p + p_0} \right)^{1+\alpha} \csc \pi\alpha,$$

where we have used the result

$$\int_0^1 d\zeta \left(\frac{\zeta}{1 - \zeta} \right)^{1-\alpha} = \pi(1 - \alpha) \csc \pi\alpha.$$

Deforming the inversion contour round the branch point and on either side of a cut from there to $-\infty$ along the real axis, we integrate by parts once to obtain the factor $(p + p_0)^{-\alpha}$ in the integrand; i.e.,

$$n = \alpha^{-1} \int_{-\infty}^{-p_0} (-p - p_0)^{-\alpha} \frac{dg}{dp} dp,$$

where $g(p) \equiv \alpha(1 - \alpha)\, p_0^2 \,(-p)^{-2+\alpha} \exp(p\tau)$. Then (6) follows immediately.

REFERENCES

Ambartsumian, V. A. 1937, *Astron. Zh.*, **14**, 207.
Cruz-González, C., and Poveda, A. 1971, *Ap. and Space Sci.*, **13**, 335.
Heggie, D. C. 1975*a*, *M.N.R.A.S.*, **173**, 729.
Heggie, D. C. 1975*b*, in *The Dynamics of Stellar Systems*, ed. A. Hayli (Dordrecht: D. Reidel).
Öpik, E. J. 1973, *Irish Astron. J.*, **11**, 125.

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DISCUSSION

King: I believe that Heggie and I are very much in agreement, point by point. (Naturally, I hope that this means that we are both right.) To label the correspondences: Heggie's t_1 corresponds to what I have called Ambartzumian's break-up time; his t_0 corresponds to what I referred to as the smaller probability that a single encounter will break up a binary; and his t_2 corresponds to the total tidal effect, to which I referred earlier in answering Huang's question. The place where Heggie and I differ is that for smaller changes in energy he retains the rather intractable Boltzmann integral, whereas I prefer the more workable Fokker-Planck equation, even in the face of the fact that it is not a very close approximation.

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