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# ON EQUILIBRIUM FIGURES FOR IDEAL FLUIDS IN THE FORM OF CONFOCAL SPHEROIDS ROTATING WITH COMMON AND DIFFERENT ANGULAR VELOCITIES

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RESUMEN

Para una masa fluída que consiste de dos esferoides *confocales* de diferentes densidades, se demuestra, primero, la inexistencia de figuras de equilibrio si ambos esferoides giran con velocidad angular común y, segundo, la existencia de figuras de equilibrio si giran con diferentes velocidades angulares. Se asume que el fluído propuesto es incompresible, auto-gravitante, libre de presión externa y que cada esferoide del modelo es homogeneo *per se*.

ABSTRACT

For a fluid mass consisting of two *confocal* spheroids each one with different density, we demonstrate, firstly, the non-existence of equilibrium figures if both spheroids rotate with a common angular velocity and, secondly, the existence of equilibrium figures if they rotate with different angular velocities. The fluid is to be considered incompressible, self-gravitating, free from any external pressure and that each spheroid of the model is homogeneous *per se*.

*Key words:* **HYDRODYNAMICS**

## I. INTRODUCTION

Incompressible self-graviting homogeneous fluids rotating with constant angular velocity adopt, in general, the ellipsoidal form, the rotation axis being coincident with the shortest ellipsoidal axis (Lyttleton 1951). This was demonstrated by MacLaurin for rotating spheroidal (a special case of ellipsoidal forms) figures for any value of their angular momentum, and by Jacobi (a hundred years later) for rotating ellipsoidal figures, only for values of their angular momentum exceeding a certain finite number.

In the present paper, we inquire about equilibrium figures for a composite fluid body (rather than a single fluid body as treated by MacLaurin or Jacobi) but each of whose parts is also an ideal fluid of different density.

For this purpose, following MacLaurin's rather than Jacobi's forms, we constructed a model consisting of two spheroids confocal to each other and free from any external pressure. We examine two cases: one, in which both spheroids of the model rotate with a common angular velocity ($\omega$) and, the other, in which they rotate with different angular velocities. Each spheroid of the model is assumed homogeneous *per se*.

Thus, our model can be visualized as consisting of a "nucleus", of density $\rho_n$, surrounded by an "atmosphere", of density $\rho_a$, confocal to the nucleus. Firstly, we consider the $\omega_n = \omega_a$ case and demonstrate the non-existence of equilibrium figures. Lastly, we turn to the $\omega_n \neq \omega_a$ case and demonstrate the existence of equilibrium figures. In both cases, $\rho_n \neq \rho_a$.

In the next section, we obtain the equilibrium conditions our model must fulfill, for either the $\omega_n = \omega_a$ or the $\omega_n \neq \omega_a$ case.

## II. EQUILIBRIUM CONDITIONS

For a fluid at rest, under the action of conservative forces, the equilibrium condition follows from Euler's equation (Milne-Thomson 1968): one needs only to take zero for the velocity at each point of the fluid in that equation, and integrate it over its volume to obtain

$$P = \rho\, B + \text{constant} \quad , \tag{1}$$

where P is the pressure, $\rho$ the density and B the gravitational potential. If the fluid is rotating, as we will assume throughout this paper, then equation (1) must be modified into

$$P/\rho = B + \tfrac{1}{2}\omega^2 (x_1^2 + x_2^2) + \text{constant} \quad , \tag{2}$$

where $\vec{\omega} = \omega \hat{k}$, and $\hat{k}$ is the unit vector along the $x_3$ axis. In particular, equation (2) becomes

$$B + \tfrac{1}{2}\omega^2(x_1^2+x_2^2) + \text{constant} = 0 \quad , \qquad (3)$$

on the surface of the fluid, because we assumed that P = 0 at each point on it.

Denoting the left hand side of equation (3) by $\phi$ and the surface equation by $f(x_1,x_2,x_3)=0$, then the following ratios must be satisfied (Lyttleton 1951)

$$\phi_{x_1}/f_{x_1} = \phi_{x_2}/f_{x_2} = \phi_{x_3}/f_{x_3} \quad , \qquad (4)$$

where the subscripts stand for partial derivatives.

Each point of the surface must therefore satisfy equation (4). Another physical condition our model must satisfy is

$$P_n = P_a \quad , \qquad (5)$$

deduced from the continuity of the pressure across the surface that divides the nucleus and the atmosphere; to be certain, equation (5) holds only if matter does not flow, in either sense, across that boundary surface and if surface tension is disregarded, as we will assume.

In the next section we present the well known expressions for the potential of homogeneous ellipsoids, which will be particularized for spheroids later on.

a) *Potential of Homogeneous Ellipsoids*

The potential of a homogeneous ellipsoid at an internal point (Chandrasekhar 1969) is given by the integral expression

$$B = \pi G \rho\, a_1\, a_2\, a_3 \int_0^\infty \frac{du}{\Delta}\left(1 - \frac{x_i^2}{a_i^2+u}\right) , \qquad (6)$$

where $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$;

$a_1$, $a_2$, and $a_3$ being the ellipsoid semiaxes, G the gravitational constant and again, the density. Expression (6) is valid also for an external point if one replaces the lower limit of the integral by $\lambda$, the ellipsoidal coordinate of the considered point, that is, the positive root of

$$\Sigma \frac{x_i^2}{a_i^2+\lambda} = 1 \quad . \qquad (7)$$

b) *Construction of the Model*

Because the potential of homogeneous ellipsoids is well known (equation 6) and since such an expression is easily integrable in the case of spheroids, these forms were chosen for the nucleus and atmosphere of our fluid body. Furthermore, these spheroids were assumed confocal (as it will be justified later on).

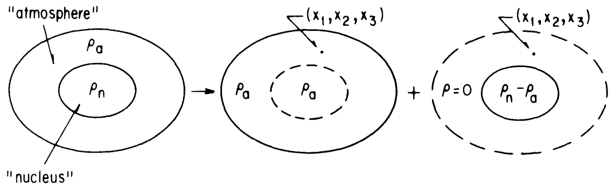


Fig. 1. The artifice used for obtaining the potentials of the nucleus and the atmosphere, respectively.

With the help of Figure 1, which simulates our model, we can deduce the potentials $B_n$, at each point of the nucleus and $B_a$, at each point of the atmosphere.

To deduce $B_a$ we proceed as follows: we consider first a completely homogeneous ellipsoid of density $\rho_a$ and calculate the potential at an internal point $(x_1,x_2,x_3)$; inside this homogeneous ellipsoid we place a fictitious homogeneous ellipsoidal nucleus of density $\rho_n-\rho_a$, to compensate the excess density of the former calculation, and calculate the potential at the same point $(x_1,x_2,x_3)$, which is now an external point.

Adding up both potentials, we obtain

$$B_a = \pi G \rho_a a_{a_1} a_{a_2} a_{a_3} \int_0^\infty \frac{du}{\Delta}\left(1 - \Sigma \frac{x_i^2}{a_{a_i}^2+u}\right)$$

$$+ (\rho_n-\rho_a)\, a_{n_1} a_{n_2} a_{n_3} \int_\lambda^\infty \frac{du}{\Delta}\left(1 - \Sigma \frac{x_i^2}{a_{n_i}^2+u}\right) \qquad (8)$$

where $a_a$ and $a_n$ stand for the atmosphere and the nucleus semiaxes respectively. The integrals in equation (8) are easily evaluated if we restrict to the case where $a_{a_1} = a_{a_2}$, $a_{n_1} = a_{n_2}$, that is, to spheroidal figures. In this case, integrating equation (8) (MacMillan 1958), we have

$$B_a = \frac{2\pi \rho_a a_{a_1}^2 a_{a_3}}{(a_{a_1}^2 - a_{a_3}^2)^{1/2}} \left(1 - \frac{x_1^2 + x_2^2 - 2x_3^2}{2(a_{a_1}^2 - a_{a_3}^2)}\right)$$

$$\sin^{-1}\left(\frac{a_{a_1}^2 - a_{a_3}^2}{a_{a_1}^2}\right)^{1/2} +$$

$$+ \frac{\pi\rho_a a_{a_3}^2}{a_{a_1}^2 - a_{a_3}^2}\left(x_1^2 + x_2^2\right) - \frac{\pi\rho_a a_{a_1}^2}{a_{a_1}^2 - a_{a_3}^2}\, 2x_3^2 +$$

$$+ \frac{2\pi(\rho_n - \rho_a)\, a_{n_1}^2 a_{n_3}^2}{(a_{n_1}^2 - a_{n_3}^2)^{1/2}} \left(1 - \frac{x_1^2 + x_2^2 - 2x_3^2}{2(a_{n_1}^2 - a_{n_3}^2)}\right)$$

$$\sin^{-1}\left(\frac{a_{n_1}^2 - a_{n_3}^2}{a_{n_1}^2 + \lambda}\right)^{1/2} +$$

$$+ \frac{(\rho_n - \rho_a) a_{n_1}^2 a_{n_3}^2 (a_{n_3}^2 + \lambda)^{1/2}}{a_{n_1}^2 - a_{n_3}^2} \left(\frac{x_1^2 + x_2^2}{a_{n_1}^2 + \lambda}\right)$$

$$- \frac{\pi(\rho_n - \rho_a) a_{n_1}^2 a_{n_3}}{a_{n_1}^2 - a_{n_3}^2} \left(\frac{2x_3^2}{(a_{n_3}^2 + \lambda)^{1/2}}\right) \qquad (9)$$

By looking at Figure 1, it is clear that expression (9) for $\lambda = 0$, also gives the potential $B_n$, at each point in the nucleus. Now, if one tries to use expression (9) as such, one finds that it is impractical because $\lambda$ is a complicated function of $x_1, x_2, x_3$ (see equation 7) so, we must simplify further our model. To this end, we will assume that the nucleus and the atmosphere are confocal, because then, $\lambda$ would be constant all over the external surface. The confocality relations for our model are

$$a_{a_3}^2 = a_{n_3}^2 + \lambda,$$

$$a_{a_1}^2 = a_{n_1}^2 + \lambda \;,$$

$$a_{a_1}^2 - a_{a_3}^2 = a_{a_1}^2 e_a^2,$$

$$a_{n_1}^2 - a_{n_3}^2 = a_{n_1}^2 \; e_n^2 \;. \qquad (10)$$

or $e_n/e_a = a_{a_1}/a_{n_1} > 1$, where $e_n$ and $e_a$ stand for the eccentricities of the nucleus and the atmosphere, respectively. So, in this work we will always have $e_n > e_a$.

### c) *Equilibrium Conditions for Rigid Body Rotation (Case $\omega_n = \omega_a$)*

Let us imagine that our body remains isolated from any external pressure. Equation (3) expresses this fact for an arbitrary fluid. For our model, equation (3) becomes

$$\phi + \text{constant} = B_a + \tfrac{1}{2}\omega^2 (x_1^2 + x_2^2) + \text{constant} \;, \qquad (11)$$

where $B_a$, given by equation (9), is a complicated function of $x_1, x_2, x_3$, unless we restrict ourselves to confocal spheroids, as we will do it in the rest of the paper.

We now use the surface equation of the atmosphere

$$f_a = \frac{x_1^2}{a_{a_1}^2} + \frac{x_2^2}{a_{a_1}^2} + \frac{x_3^2}{a_{a_3}^2} - 1 \;, \qquad (12)$$

together with equation (10) and (11) to obtain (after making these expressions to fit equation 4)

$$\Omega^2 = \left[\frac{(1-e_a^2)^{1/2}(3-2e_a^2)}{e_a^3} + \frac{(1-e_n^2)^{1/2}(3-2e_a^2)}{e_n^3}\,\epsilon\right] \sin^{-1} e_a$$

$$- \frac{3(1-e_a^2)}{e_a^2} - \frac{3e_a(1-e_n^2)^{1/2}(1-e_a^2)^{1/2}}{e_n^3}\,\epsilon \;, \qquad (13)$$

$$\text{where } \Omega^2 \equiv \frac{\omega^2}{2\pi G\rho_a}$$

$$\text{and } \epsilon \equiv \frac{\rho_n - \rho_a}{\rho_a}$$

This equation is a relation for $\omega$ containing the parameters that give the geometrical and physical structure of our model. Note that if one sets $\rho_n = \rho_a$ in equation(13), one recovers the familiar relationship for homogeneous spheroids (Chandrasekhar 1969).

Expression (13) is a relationship between $\omega$, $e_a$ and $e_n$, and it represents a two parametric family of equilibrium figures for a given $\epsilon$, that is, for a given density ratio $\rho_n/\rho_a$. For fixed, $\omega$, we have then an infinite number of equilibrium figures, each with their own eccentricity values $e_n$, $e_a$: the same density distribution and the same angular velocity give figures with different oblateness values of the nucleus and the atmosphere. This ambiguity can be removed by using the other condition our model must fulfill, that is, the continuity of pressure at the surface that divides the nucleus and the atmosphere; this is equation (5) and can be written as

$$\phi \equiv \rho_a B_a + \tfrac{1}{2}\, \rho_a (x_1^2 + x_2^2)\omega^2 - \rho_n B_n$$

$$- \tfrac{1}{2}\, \rho_n (x_1^2 + x_2^2)\omega^2 + \text{constant} \quad . \qquad (14)$$

Using the surface equation of the nucleus

$$f_n = \frac{x_1^2}{a_{n_1}^2} + \frac{x_2^2}{a_{n_1}^2} + \frac{x_3^2}{a_{n_3}^2} - 1 \qquad (15)$$

together with equations (10) and (14), and the fact that

$$\rho_a^2 - \rho_n \rho_a = -\epsilon \rho_a^2 \qquad (16)$$

we obtain (after making these expressions to fit equation 4).

$$\Omega^2 = \frac{(1-e_a^2)^{1/2}\,(3-2e_n^2)}{e_a^3} \sin^{-1} e_a - \frac{3-2e_n^2 - e_a^2}{e_a^2} +$$

$$+ \frac{(1-e_n^2)^{1/2}(3-2e_n^2)}{e_n^3}\, \epsilon \sin^{-1} e_n - \frac{3(1-e_n^2)}{e_n^2}\, \epsilon \quad . \qquad (17)$$

Combining equation (13) (deduced from the fact that no external pressure acts on the body) and equation (17) (deduced from the continuity of the pressure across the surface that divides the nucleus and the atmosphere) we obtain

$$\epsilon = -\left\{ (e_n/e_a)^3\, 2(e_n^2 - e_a^2) \quad \left[ (1-e_a^2)^{1/2} \sin^{-1} e_a - e_a \right] \right\}$$

$$\Big/ \left\{ (1-e_n^2)^{1/2} \left[ (3-2e_a^2) \sin^{-1} e_a - (3-2e_n^2) \sin^{-1} e_n \right. \right.$$

$$\left. \left. + 3e_n (1-e_n^2)^{1/2} - 3e_a (1-e_a^2)^{1/2} \right] \right\}, \qquad (18)$$

which gives $\epsilon$ as a function of both $e_n$ and $e_a$.

### c) *Non-existence of Equilibrium Figures* (case $\omega_n = \omega_a$)

Relation (18) is decisive to dilucidate whether or not equilibrium figures can exist for our model because, if we can show that $\epsilon < -1$, then no equilibrium figures are possible. This assertion comes from the fact that $\epsilon = \rho_n/\rho_a - 1$ and negative densities are not allowed. If, however, $-1 \leqslant \epsilon \leqslant 0$, then $\rho_a > \rho_n > 0$, but this case is not a current one.

Table 1 gives the $\epsilon$ values as calculated from equation (18) for case $\omega_n = \omega_a$ when $e_a$ and $e_n$ are varied through an increasing set of values, and conformed the basis to reach our conclusion, that is, that no equilibrium figures exist under these circumstances. A typical column gives, for fixed $e_n$, the calculated $\epsilon$ values when $e_a$ is varied through a set of increasing values. Each column shows two trends: one, in which $e_n > e_a$ and $\epsilon < -1$ and the other one, in which $e_a > e_n$ and $-1 < \epsilon < 0$. This last case is not allowed since from equation (10) we deduced that $e_n > e_a$. Figure 2 is the graphic representation of $\epsilon$, as a function of $e_a$, for some $e_n$ values.

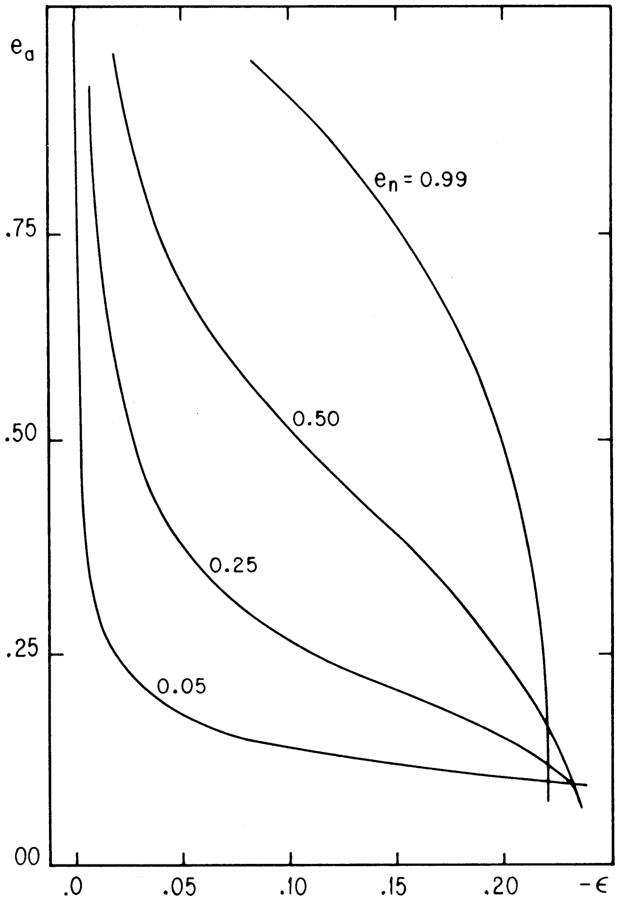


Fig. 2. Plot of $\epsilon$ vs. $e_a$ for different values of $e_n$.

### d) *Equilibrium Conditions for Rigid Body Rotation* (case $\omega_n \neq \omega_a$)

We now wish to treat the case $\omega_n \neq \omega_a$. For this purpose we return momentarily to §II*c* of the first analyzed case and make the corresponding considerations.

In this case, equation (3) becomes

$$\phi + \text{constant} = B_a + \tfrac{1}{2}\, \omega_a^2\, (x_1^2 + x_2^2) + \text{constant} \quad , \qquad (19)$$

where this equation differs from equation (11) of the former case, only in that a subscript is now attached to the angular velocity. We now make use of equation (12) for the surface equation of the atmosphere, together with equations (19) and (10) to obtain (after making these expressions to fit equation 4).

TABLE 1

DEPENDENCE OF $-\epsilon$ ON THE PARAMETERS $e_a$, $e_n$.
TAKEN FROM EQUATION (18) FOR THE CASE $\omega_n = \omega_a$

| | $e_n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_a$ | .05 | .10 | .15 | .20 | .25 | .30 | .35 | .40 | .45 | .50 |
| | | 1.934 | 2.226 | 2.335 | 2.382 | 2.402 | 2.410 | 2.408 | 2.403 | 2.393 |
| .10 | 0.242 | | 1.599 | 1.931 | 2.111 | 2.211 | 2.270 | 2.303 | 2.321 | 2.329 |
| .15 | 0.0823 | 0.473 | | 1.434 | 1.731 | 1.926 | 2.052 | 2.134 | 2.188 | 2.223 |
| .20 | 0.0363 | 0.240 | 0.604 | | 1.339 | 1.597 | 1.785 | 1.919 | 2.013 | 2.081 |
| .25 | 0.0189 | 0.134 | 0.372 | 0.6837 | | 1.278 | 1.503 | 1.678 | 1.811 | 1.911 |
| .30 | 0.0110 | 0.0812 | 0.239 | 0.4705 | 0.737 | | 1.235 | 1.439 | 1.595 | 1.724 |
| .35 | 0.0069 | 0.0522 | 0.160 | 0.3297 | 0.544 | 0.7748 | | 1.204 | 1.382 | 1.531 |
| .40 | 0.0046 | 0.0353 | 0.111 | 0.2363 | 0.405 | 0.5994 | 0.8029 | | 1.181 | 1.341 |
| .45 | 0.0032 | 0.0248 | 0.0792 | 0.1731 | 0.3048 | 0.4655 | 0.6428 | 0.8243 | | 1.163 |
| .50 | 0.0023 | 0.0180 | 0.0582 | 0.1294 | 0.2326 | 0.3638 | 0.5151 | 0.6772 | 0.8412 | |
| .55 | 0.0017 | 0.0134 | 0.0437 | 0.0984 | 0.1799 | 0.2866 | 0.4140 | 0.5558 | 0.7049 | 0.8547 |
| .60 | 0.0013 | 0.0102 | 0.0334 | 0.0761 | 0.1407 | 0.2275 | 0.3342 | 0.4565 | 0.5893 | 0.7272 |
| .65 | 0.0010 | 0.0079 | 0.0260 | 0.0595 | 0.1112 | 0.1820 | 0.2709 | 0.3754 | 0.4921 | 0.6167 |
| .70 | $7.81 \times 10^{-4}$ | 0.0061 | 0.0204 | 0.0471 | 0.0887 | 0.1465 | 0.2204 | 0.3092 | 0.4106 | 0.5215 |
| .75 | $6.17 \times 10^{-4}$ | 0.0049 | 0.0162 | 0.0376 | 0.0712 | 0.1184 | 0.1798 | 0.2549 | 0.3422 | 0.4398 |
| .80 | $4.90 \times 10^{-4}$ | 0.0039 | 0.0129 | 0.0301 | 0.0573 | 0.0960 | 0.1469 | 0.2100 | 0.2846 | 0.3695 |
| .85 | $3.90 \times 10^{-4}$ | 0.0031 | 0.0103 | 0.0242 | 0.0462 | 0.0778 | 0.1198 | 0.1724 | 0.2356 | 0.3085 |
| .90 | $3.10 \times 10^{-4}$ | 0.0025 | 0.0082 | 0.0193 | 0.0370 | 0.0626 | 0.0969 | 0.1404 | 0.1931 | 0.2548 |
| .95 | $2.42 \times 10^{-4}$ | 0.0019 | 0.0064 | 0.0151 | 0.0291 | 0.0493 | 0.0767 | 0.1117 | 0.1546 | 0.2053 |

| | $e_n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_a$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
| | 2.382 | 2.370 | 2.357 | 2.346 | 2.339 | 2.343 | 2.368 | 2.446 | 2.714 | 4.205 |
| .10 | 2.330 | 2.327 | 2.322 | 2.317 | 2.315 | 2.322 | 2.350 | 2.431 | 2.699 | 4.185 |
| .15 | 2.244 | 2.257 | 2.264 | 2.269 | 2.275 | 2.288 | 2.321 | 2.405 | 2.675 | 4.152 |
| 20 | 2.128 | 2.161 | 2.184 | 2.202 | 2.218 | 2.240 | 2.280 | 2.369 | 2.641 | 4.106 |
| .25 | 1.985 | 2.041 | 2.084 | 2.117 | 2.146 | 2.179 | 2.227 | 2.323 | 2.597 | 4.046 |
| .30 | 1.825 | 1.904 | 1.966 | 2.017 | 2.061 | 2.105 | 2.163 | 2.266 | 2.543 | 3.972 |
| .35 | 1.654 | 1.754 | 1.836 | 1.903 | 1.962 | 2.020 | 2.089 | 2.200 | 2.480 | 3.885 |
| .40 | 1.479 | 1.597 | 1.696 | 1.780 | 1.854 | 1.924 | 2.004 | 2.124 | 2.406 | 3.783 |
| .45 | 1.309 | 1.439 | 1.551 | 1.649 | 1.737 | 1.820 | 1.911 | 2.039 | 2.324 | 3.668 |
| .50 | 1.149 | 1.284 | 1.406 | 1.516 | 1.615 | 1.709 | 1.810 | 1.946 | 2.232 | 3.538 |
| .55 | | 1.137 | 1.264 | 1.381 | 1.490 | 1.594 | 1.704 | 1.846 | 2.132 | 3.394 |
| .60 | 0.8654 | | 1.128 | 1.249 | 1.364 | 1.475 | 1.592 | 1.740 | 2.024 | 3.253 |
| .65 | 0.7452 | 0.8739 | | 1.122 | 1.239 | 1.355 | 1.477 | 1.628 | 1.908 | 3.062 |
| .70 | 0.6388 | 0.7593 | 0.8803 | | 1.118 | 1.236 | 1.360 | 1.512 | 1.785 | 2.873 |
| .75 | 0.5452 | 0.6560 | 0.7698 | 0.8847 | | 1.117 | 1.241 | 1.392 | 1.654 | 2.667 |
| .80 | 0.4629 | 0.5631 | 0.6680 | 0.7761 | 0.8864 | | 1.121 | 1.268 | 1.515 | 2.440 |
| .85 | 0.3902 | 0.4752 | 0.5741 | 0.6736 | 0.7768 | 0.8843 | | 1.138 | 1.365 | 2.187 |
| .90 | 0.3249 | 0.4024 | 0.4863 | 0.5755 | 0.6694 | 0.7681 | 0.8745 | | 1.199 | 1.892 |
| .95 | 0.2636 | 0.3288 | 0.4003 | 0.4772 | 0.5587 | 0.6448 | 0.7368 | 0.8424 | | 1.514 |

$$\Omega_a^2 \equiv \frac{(1-e_a^2)^{1/2}(3-2e_a^2)}{e_a^3} \sin^{-1} e_a + \frac{(1-e_n^2)^{1/2}(3-2e_a^2)}{e_n^3}$$

$$\times\ \epsilon \sin^{-1} e_a - \frac{3(1-e_a^2)}{e_a^2} - \frac{3e_a(1-e_n^2)^{1/2}(1-e_a^2)^{1/2}}{e_{nn}^3} \epsilon \quad (20)$$

$$\text{where } \Omega_a^2 \equiv \frac{\omega_a^2}{2\pi G \rho_a} \quad \text{and} \quad \epsilon \equiv \frac{\rho_n - \rho_a}{\rho_a}$$

Expression (20) (which is identical to equation 13) is a relationship between $\omega_a$, $e_a$ and $e_n$ and it represents again a two parametric family of equilibrium figures for a given $\epsilon$, that is, for a given density ratio $\rho_n/\rho_a$. But since we inquire about equilibrium figures with $\omega_n \neq \omega_a$, we must treat $\omega_n$ on the same footing as $\omega_a$.

We can obtain another independent expression, with the same parameters as those involved in equation (13), and with $\omega_n$ as well, if we now use the continuity condition for the pressure across the surface that divides the nucleus and the atmosphere, which can be written as (see equation 5).

$$\phi \equiv \rho_a B_a + \tfrac{1}{2}\rho_a(x_1^2 + x_2^2)\omega_a^2 - \rho_n B_n$$

$$- \tfrac{1}{2}\rho_n(x_1^2 + x_2^2)\omega_n^2 + \text{constant} , \quad (21)$$

where this equation differs from equation (14) of the former case, only in that subscripts are now attached to the angular velocities. We now make use of equation (15) for the surface equation of the nucleus, together with equations (10), (16) and (21) and taking into account equation (4) to obtain

$$\Omega_n^2(\epsilon + 1) - \Omega_a^2 = \frac{(1-e_a^2)^{1/2}(3-2e_n^2)}{e_a^3} \epsilon \sin^{-1} e_a +$$

$$+ \frac{(1-e_n^2)^{1/2}\ (3-2e_n^2)}{e_n^3} \epsilon^2 \sin^{-1} e_n$$

$$- \frac{3-e_a^2-2e_n^2}{e_a^2} \epsilon - \frac{3(1-e_n^2)}{e_n^2} \epsilon^2 . \quad (22)$$

Combining equations (20) and (22), we obtain

$$\Omega_n^2 (\epsilon + 1) = \left[\frac{(1-e_n^2)^{1/2}}{e_n^3} (3-2e_n^2) \sin^{-1} e_n\right.$$

$$\left. - \frac{3(1-e_n^2)}{e_n^2}\right] \epsilon^2 + \left[\frac{(1-e_a^2)^{1/2}}{e_a^3} (3-2e_n^2) \sin^{-1} e_a\right.$$

$$- \frac{3e_a(1-e_n^2)^{1/2}\ (1-e_a^2)^{1/2}}{e_n^3} - \frac{3-e_a^2-2e_n^2}{e_a^2}$$

$$\left. + \frac{(1-e_n^2)^{1/2}}{e_n^3} (3-2e_a^2) \sin^{-1} e_a\right] \epsilon +$$

$$+ \left[\frac{(1-e_a^2)^{1/2}}{e_a^3} (3-2e_a^2) \sin^{-1} e_a\right.$$

$$\left. - \frac{3(1-e_a^2)}{e_a^2}\right] \epsilon^0 , \quad (23)$$

which gives $\omega_n$ as a function of $\epsilon$, $e_n$ and $e_a$.

e) *Existence of Equilibrium Figures*
(case $\omega_n \neq \omega_a$)

Relation (23) is decisive to dilucidate whether or not equilibrium figures can exist, because if we can show that the right hand side of equation (23) is positive, then such equation will be soluble, since its left hand side is positive definite (because $\epsilon > 0$) and there will be equilibrium figures.

Let us take the rectangular parenthesis which is factor of $\epsilon^2$ in equation (23). After factorizing, we get

$$\frac{(1-e_n^2)^{1/2}}{e_n^3} \left[(3-2e_n^2) \sin^{-1} e_n - 3e_n(1-e_n^2)^{1/2}\right] . \quad (24)$$

Let us call

$$f(e) = (3 - 2e^2) \sin^{-1} e - 3e\,(1 - e^2)^{1/2} ,$$

which becomes

$$f(y) = (3 - 2\sin^2 y)\, y - 3 \sin y \cos y,$$

after the change of variable

$$y = \sin^{-1} e.$$

TABLE 2

$\Omega_n^2, \Omega_a^2$ VALUES FOR CASE $\omega_n \neq \omega_a$, FOR DIFFERENT PARAMETERS $\epsilon$, $e_n$, $e_a$. TAKEN FROM EQUATION (23)

| $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ | $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ | $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ |
|---|---|---|---|---|---|---|---|---|
| | | | | $\epsilon = .1$ | | | | |
| | $e_a = .1$ | | | $e_a = .2$ | | | $e_a = .3$ | |
| .15 | .003557 | .002749 | .25 | .012759 | .011270 | .35 | .028021 | .025811 |
| .2 | .004623 | .002703 | .3 | .014309 | .011037 | .4 | .030003 | .025293 |
| .3 | .007768 | .002679 | .4 | .018630 | .010853 | .5 | .035433 | .024784 |
| .4 | .012198 | .002674 | .5 | .024345 | .010788 | .6 | .042439 | .024562 |
| .5 | .017903 | .002672 | .6 | .031383 | .010760 | .7 | .050861 | .024451 |
| .6 | .024884 | .002671 | .7 | .039724 | .010745 | .8 | .060630 | .024388 |
| .7 | .033139 | .002671 | .8 | .049346 | .010737 | .9 | .071640 | .024349 |
| .8 | .042656 | .002670 | .9 | .060166 | .010732 | | | |
| .9 | .053352 | .002670 | | | | | | |
| | $e_a = .4$ | | | $e_a = .5$ | | | $e_a = .6$ | |
| .45 | .049611 | .046620 | .55 | .077860 | .073995 | .65 | .113077 | .108197 |
| .5 | .052020 | .045745 | .6 | .080713 | .072685 | .7 | .116404 | .106335 |
| .6 | .058552 | .044762 | .7 | .088392 | .071070 | .8 | .125303 | .103861 |
| .7 | .066382 | .044268 | .8 | .098115 | .070163 | .9 | .136428 | .102301 |
| .8 | .076753 | .043991 | .9 | .109457 | .069592 | | | |
| .9 | .087975 | .043816 | | | | | | |
| | $e_a = .7$ | | | $e_a = .8$ | | | $e_a = .9$ | |
| .75 | .155236 | .149113 | .85 | .202644 | .194870 | .95 | .243975 | .233680 |
| .8 | .159045 | .146475 | .9 | .206711 | .190844 | | | |
| .9 | .169102 | .142653 | | | | | | |
| | | | | $\epsilon = .5$ | | | | |
| | $e_a = .1$ | | | $e_a = .2$ | | | $e_a = .3$ | |
| .15 | .006724 | .003063 | .25 | .020421 | .013441 | .35 | .042354 | .318230 |
| .2 | .011261 | .002834 | .3 | .027103 | .012274 | .4 | .051010 | .029234 |
| .3 | .024602 | .002717 | .4 | .045521 | .011354 | .5 | .074301 | .026691 |
| .4 | .043423 | .002689 | .5 | .069860 | .011031 | .6 | .104214 | .025581 |
| .5 | .067724 | .002679 | .6 | .099890 | .010809 | .7 | .140161 | .025024 |
| .6 | .097543 | .002675 | .7 | .135537 | .010819 | .8 | .181702 | .024711 |
| .7 | .132879 | .002673 | .8 | .176541 | .010778 | .9 | .227231 | .024513 |
| .8 | .173497 | .002672 | .9 | .221368 | .10753 | | | |
| .9 | .217874 | .002671 | | | | | | |
| | $e_a = .4$ | | | $e_a = .5$ | | | $e_a = .6$ | |
| .45 | .072955 | .058564 | .55 | .112700 | .093992 | .65 | .162002 | .138285 |
| .5 | .083581 | .054189 | .6 | .125380 | .087439 | .7 | .176837 | .128973 |
| .6 | .111755 | .049276 | .7 | .158604 | .079364 | .8 | .215186 | .116603 |
| .7 | .147368 | .046806 | .8 | .199974 | .074832 | .9 | .261136 | .108802 |
| .8 | .189280 | .045419 | .9 | .246722 | .071974 | | | |
| .9 | .235588 | .044545 | | | | | | |
| | $e_a = .7$ | | | $e_a = .8$ | | | $e_a = .9$ | |
| .75 | .220710 | 190882 | .85 | .285872 | .248034 | .95 | .329721 | .31748 |
| .8 | .237590 | 177695 | .9 | .303099 | .227903 | | | |
| .9 | .279621 | .158582 | | | | | | |

TABLE 2 (CONTINUED)

| $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ | $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ | $e_n$ | $\Omega_n^2$ | $\Omega_a^2$ |
|---|---|---|---|---|---|---|---|---|
| | | | | $\epsilon = 1$ | | | | |
| | $e_a = .1$ | | | $e_a = .2$ | | | $e_a = .3$ | |
| .15 | .010256 | .003456 | .25 | .029471 | .016155 | .35 | .059686 | .039339 |
| .2 | .018238 | .002999 | .3 | .041368 | .013822 | .4 | .075270 | .034169 |
| .3 | .041645 | .002765 | .4 | .073827 | .011981 | .5 | .116548 | .029074 |
| .4 | .074703 | .002708 | .5 | .116675 | .011334 | .6 | .169337 | .026856 |
| .5 | .117500 | .002689 | .6 | .169641 | .011052 | .7 | .232755 | .025740 |
| .6 | .170149 | .002680 | .7 | .232610 | .010909 | .8 | .305795 | .025114 |
| .7 | .232652 | .002676 | .8 | .304843 | .010830 | .9 | .383758 | .024719 |
| .8 | .304306 | .002673 | .9 | .381755 | .010779 | | | |
| .9 | .380542 | .002672 | | | | | | |
| | $e_a = .4$ | | | $e_a = .5$ | | | $e_a = .6$ | |
| .45 | .101501 | .073494 | .55 | .155560 | .118987 | .65 | .222344 | .175894 |
| .5 | .120804 | .064744 | .6 | .178741 | .105881 | .7 | .249586 | .157271 |
| .6 | .170985 | .054919 | .7 | .238075 | .089731 | .8 | .317836 | .132530 |
| .7 | .233903 | .049978 | .8 | .310858 | .080667 | .9 | .396433 | .116929 |
| .8 | .307498 | .047205 | .9 | .390651 | .074951 | | | |
| .9 | .386631 | .045456 | | | | | | |
| | $e_a = .7$ | | | $e_a = .8$ | | | $e_a = .9$ | |
| .75 | .301663 | .243094 | .85 | .388748 | .314490 | .95 | .450270 | .354425 |
| .8 | .332445 | .216721 | .9 | .418925 | .274227 | | | |
| .9 | .405163 | .178494 | | | | | | |

If we can show that f(e) is positive, so it will be expression (24). The derivative of f(y) is

$$f'(y) = 4 \text{ sen } y \cos y (\tan y - y)$$

and since $(\tan y - y) > 0$ in the interval $o \leqslant y \leqslant \pi/2$, $f'(y)$ is positive and so f(e) is an increasing function (since $0 \leqslant e \leqslant 1$). Besides, $f(e) = 0$ for $e = 0$, so that f(e) is also positive. Similar reasoning for the rest of the rectangular parentheses of equation (23) show that they all are positive. This reasoning was the basis to reach our conclusion, that is, that equilibrium figures do exist under these circumstances.

Table 2 for case $\omega_n \neq \omega_a$ gives, for some values of $\epsilon$, the values of $\Omega_a^2$ and $\Omega_n^2$ as calculated from equations (20) and (23), respectively, and confirm our deduction. A typical section give, for fixed values of $\epsilon$ and $e_a$, the values of $\Omega_n^2$ and $\Omega_a^2$ when $e_n$ is varied through an increasing set of values.

From equations (20) and (22) we can see that, as $\epsilon \to 0$, $\Omega_n^2 \to \Omega_a^2$, meaning that equilibrium figures exist if our model is homogeneous and rotates with a single constant angular velocity. If $\epsilon \to -1$, $\omega$ $\Omega_n^2 \to -\infty$. The range $-1 \leqslant \epsilon \leqslant 0$ corresponds to the case $\rho_a > \rho_n > 0$, this case is not a current one (see Figure 3).

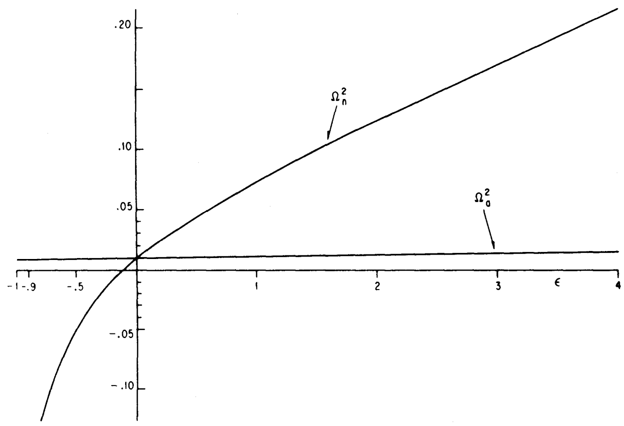


Fig. 3. Plot of $\epsilon$ vs. $\Omega^2$ showing the general trends followed by $\Omega_n^2$ and $\Omega_a^2$, as outlined in the text (for $e_n = 0.4$, $e_a = 0.2$).

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