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ON ACCRETION DISKS AND METRIC THEORIES OF GRAVITATION

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RESUMEN

La luminosidad de un disco de acreción alrededor de un objeto compacto está dada por: $L = (1 - E_0)\dot{M}c^2$, donde $\dot{M}$ es la masa por unidad de tiempo que entra al disco y E_0 es la energía en la última órbita circular estable vista desde el infinito, dividido por la energía en reposo. Se calcula esta energía y la frecuencia máxima vista desde el infinito ν_m usando la métrica estática con simetría esférica más general. Luego se usan las fórmulas obtenidas para hacer cálculos con métricas particulares, distintas de la de Schwarzschild, para ver como varían E_0 y ν_m con la teoría de gravitación utilizada.

Finalmente se extienden los resultados con la métrica estacionaria más general para tener en cuenta los efectos de rotación del objeto compacto.

ABSTRACT

The luminosity of an accretion disk around a compact object is: $L = (1 - E_0)\dot{M}c^2$, with $\dot{M}$ being the mass per unit time entering the disk and E_0 the energy as seen at infinity per rest energy at the last stable circular orbit. This energy and the maximum orbital frequency ν_m are computed using the most general metric with spherical symmetry. The computations taking some particular metrics, different from the Schwarzschild one, are then carried out to show the dependence of E_0 and ν_m on the theory of gravitation used. Finally the analysis is extended using the most general stationary metric to take into account the rotation of the compact object.

Key words: ACCRETION DISKS – GRAVITATION THEORIES

I. INTRODUCTION

The best known model of galactic hard X-ray sources is a binary stellar system made up of a normal star transferring matter onto its companion star, which is a compact object. This matter, falling inward in quasi-circular orbits, will form an accretion disk which will emit the observed X rays.

The gas will have quasi-circular orbits because every elliptical component would be rapidly damped by the action of gas in the neighboring orbits.

The gas will acquire a small inward velocity toward the compact object because of the action of viscosity torques transferring energy and angular momentum from the center outwards in the disk.

The friction due to viscosity will generate heat, which is radiated away through the disk surfaces. This energy is supplied by the loss of the total energy of the gas, while going through the disk, down to the last stable circular orbit. After this the gas would fall almost without radiating (Stroeger 1980).

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Using Schwarzschild's metric (see Bardeen *et al.* 1972 also for Kerr metric), the last stable circular orbit has an r coordinate $r_0 = 6m$, where $m = GM/c^2$ and M is the mass of the source. At this r_0 the energy "at infinity" per rest energy is $E_0 = (8/9)^{1/2}$. If we take $E = 1$ at the external radius of the disk and a steady flux of matter (or its temporal average) the total luminosity of the accretion disk will be:

$$L = (1 - E_0)\dot{M}c^2 \quad , \tag{1}$$

with $\dot{M}$ = mass per unit time entering the disk. The equation (1) is obtained from the fact that: $\dot{M}c^2$ is the energy per unit time entering the disk and $(1 - E_0)$ is the efficiency of transforming this energy into heat by the action of viscosity; afterwards this heat is radiated through the disk surfaces.

The structure of accretion disks has been studied with Newtonian theory (Shakura and Sunyaev 1973) and with General Relativity (Novikov and Thorne 1973). For further information on accretion disks see the reviews by Pringle (1981), Verbunt (1982), and Hayakawa (1985).

II. METRIC FOR A STATIC, SPHERICAL SYSTEM

It can be proved that the more general static metric with spherical symmetry can be written (Misner *et al.* 1973) as follows:

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - H(r)d\omega^2 \ , \qquad (2)$$

with $d\omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

Still we have the freedom of choosing H(r) as a function of r, for example:

$$H(r) = -r^2 \ . \qquad (2')$$

From:

$$\delta \int (ds/d\lambda)^2 \, d\lambda^2 = 0, \qquad (3)$$

we will have the equations of motion (Sarmiento G. 1982):

$$\theta = \text{constant} = \pi/2 \ , \qquad (3')$$

(because the problem is isotropic), then:

$$B(r)\, dt/d\lambda = \text{constant} = E \ , \qquad (3'')$$

$$H(r)\, d\phi/d\lambda = \text{constant} = J \ , \qquad (3''')$$

$$(ds/d\lambda)^2 = \text{constant} = -1 \ , \qquad (3'''')$$

(choosing λ properly). Substituting these in (2) we have:

$$(ds/d\lambda)^2 = B(E/B)^2 - A(dr/d\lambda)^2 - H(J/H)^2 = -1 \ .$$

Solving it for: $(dr/d\lambda)^2$ we obtain:

$$(dr/d\lambda)^2 = (E^2/B - J^2/H + 1)/A = -V_{eff} + E^2 \ ;$$

now we can use the effective potential method, as it was done in Misner *et al.* (1973, box 25.6).

In a circular orbit:

$$dr/d\lambda = 0 \quad \Rightarrow \quad E^2/B - J^2/H + 1 = 0 \ . \qquad (4)$$

If we want it to be stable, V_{eff} must be a minimum.

According to these conditions we have:

$$\partial V_{eff}/\partial r = 0$$

$$-A'[E^2/B - J^2/H + 1]/A^2 + [-E^2B'/B^2 + J^2H'/H^2]/A = 0 \ , \qquad (5)$$

where $' = d/dr$.

It will be a minimum if $\partial^2 V_{eff}/\partial r^2 > 0$; then the last stable circular orbit will be given by:

$$\partial^2 V_{eff}/\partial r^2 = 0$$

$$-A''[E^2/B - J^2/H + 1]/A^2 + 2(A')^2 [E^2/B - J^2/H + 1]/A^3 - 2A'[-E^2B'/B^2 + J^2H'/H^2]/A^2 + [-E^2B''/B^2 + 2E^2(B')^2/B^3 + J^2H''/H^2 - 2J^2(H')^2/H^3]/A = 0 \ . \qquad (6)$$

From (4), (5) and (6) we have three equations to solve for (E, J and r) at the last stable circular orbit.

From (4) and (5):

$$E^2 = -B^2/(B - B'H/H') \ , \qquad (7)$$

$$J^2 = -B'H^2/[H'(B - B'H/H')] \ . \qquad (8)$$

Substituting these in (6):

$$H'H[-B''B + 2(B')^2] + BB'[H''H - 2(H')^2] = 0 \ , \qquad (9)$$

where substituting B(r) and H(r) we find r_0 and going back to (7) and (8) we find E_0 and J_0 (note that these results do not depend on A(r)).

It is also of interest to obtain the maximum orbital frequency as seen from infinity ν_m since this has been proposed to be measured as a test for gravitation (Novikov and Thorne 1973).

The frequency "at infinity" is $\nu = d\phi/dt$. Then, from equations (3):

$$\nu = (J/H)(B/E) \ , \qquad (10)$$

and from (7) and (8) we obtain:

$$\nu^2 = B'/H'$$

valid for every circular orbit.

If B(r) and H(r) are given we must search for the maximum in the range $r_0 < r < \infty$.

We will turn now to particular computations.

III. COMPUTATIONS

As examples of applications of these formulae we will take several metrics, whose solutions are according to other theories of gravitation, different from Schwarzschild's General Relativity solution.

1) Scalar-Tensor Theories: The scalar-tensor theories have the Schwarzschild metric as a solution (Will 1981), then if the metric's source were a black hole we would obtain the following values from General Relativity:

$$r_0 = 6m, E_0 = (8/9)^{1/2}, P_{min} = 2\pi/\nu_m = 2\pi 6(6)^{1/2} m = 5 \times 10^{-4} (M/M_\odot) \text{ sec}$$

2) Rosen Theory: (Will 1981)

$$ds^2 = -\exp(-2M_\phi/r)dt^2 + \exp(2M_\Lambda/r)(dr^2 + r^2 d\omega^2).$$

There is no horizon in this metric, only a naked singularity at $r = 0$.

$$E^2 = (1 - M_\Lambda/r)\exp(-2M_\phi/r)/(1 - M_\phi/r - M_\Lambda/r)$$

$$(M\nu)^2 = (M_\phi/r)^3 \exp(-2(M_\phi + M_\Lambda)/r)/(1 - M_\Lambda/r)$$

$$r_0 = M_\phi + 2M_\Lambda + ((M_\phi + M_\Lambda)^2 + M_\Lambda^2)^{1/2}.$$

In the post newtonian limit $M_\Lambda/M_\phi = \gamma$ and from time delay $|\gamma - 1| < 10^{-3}$ (Will 1984). We then take

$$r_0/M_\phi = 3 + (5)^{1/2}.$$

From where:

$$1 - E_0 = .054787$$

$$\Rightarrow \Delta L/L = (L_R - L_{6R})/L_{6R} = -4.2\%$$

and

$$P_{min} = 5 \times 10^{-4} (M_\phi/M_\odot) \text{ sec}$$

$$\Rightarrow P_R/P_{6R} = \nu_m^{6R}/\nu_m^R = 1.074 .$$

In the Theory of Chang and Johnson 1980, $M_\phi = M_\Lambda$ in such case all the previous results are exact.

3) Lightman-Lee Theory (Lightman *et al.* 1979, p. 205):

$$ds^2 = -[(1 - u/2)^2/(1 + u/2)^2]dt^2 + [(1 - u/2)^2/(1 - 3u/2)^2](dr^2 + r^2 d\omega^2),$$

with $u = m/r$.

This metric has an event horizon at $r = m/2$.

$$E^2 = (1 - 4u + 4u^2 - 3u^3/2 + 3u^4/16)/(1 - 3u - u^2 + 3u^3/4 + 3u^4/16),$$

$$(m\nu)^2 = (1 - 3u/2)^3 u^3/[(1 + u/2)^3 (1 - 3u + 3u^2/4)].$$

Solving:

$$1 - 19u/2 + 17u^2 - 13u^3/2 - 33u^4/16 - 27u^5/32 = 0$$

yields $r_0/m = 7.2975$;

then $1 - E_0 = .0436 \Rightarrow \Delta L/L = -24\%$,

$$P_{min} = 7.5 \times 10^{-4} (M/M_\odot) \text{ sec} \Rightarrow P_m^{LL}/P_m^{GR} = 1.63 .$$

One might think that these results, close to the General Relativity values are due to the fact that the metrics used have the same post Newtonian limit as the Schwarzschild one. In that case, these experiments would not add anything new. But the next example shows that this assumption is wrong.

4) With $H(r) = -r^2$ eqs. (7), (8) and (9) take the following form:

$$E^2 = -B/(B - rB'/2), \quad (7')$$

$$J^2 = r^3 B'/(2B - rB'), \quad (8')$$

$$-B'' + 2(B')^2/B - 3B'/r = 0 . \quad (9')$$

We will study:

$$B(r) = -(1 - 2m/r + 2(\beta - \gamma)(m/r)^2 + \Omega(m/r)^3),$$

where β and γ are the P-P-N parameters (Weinberg 1972) while a new parameter Ω is introduced.

By measurements in the solar system (Will 1984) $|2(\beta-\gamma)| < 10^{-2}$ and as $m/r \sim 1/6$ we have:

$B(r) = -(1 - 2m/r + \Omega(m/r)^3)$ valid for every m/r.

Now, with $u = m/r$:

$$E^2 = [1 - 4u + 4u^2 + 2\Omega u^3 - 4\Omega u^4 + \Omega^2 u^6]/[1 - 3u + 2.5\Omega u^3], \quad (11)$$

$$(m\nu)^2 = u^3 - 1.5\Omega u^5, \quad (12)$$

and the equation for $u_0 = m/r_0$ becomes:

$$\Omega = [20u_0 + 3 \pm (-320u_0^2 + 240u_0 + 9)^{1/2}]/(30u_0)$$

from where we find[2]: $r_0(\Omega)$ and $E_0(\Omega)$ from (11).

Figure 1 shows in percentage the relative variation of the luminosity with respect to the General Relativity value ($\Omega = 0$) as a function of Ω.

From (1) we have:

$$\Delta L/L = [L(\Omega) - L(0)]/L(0) = [E_0(0) - E_0(\Omega)]/[1 - E_0(0)]. \quad (13)$$

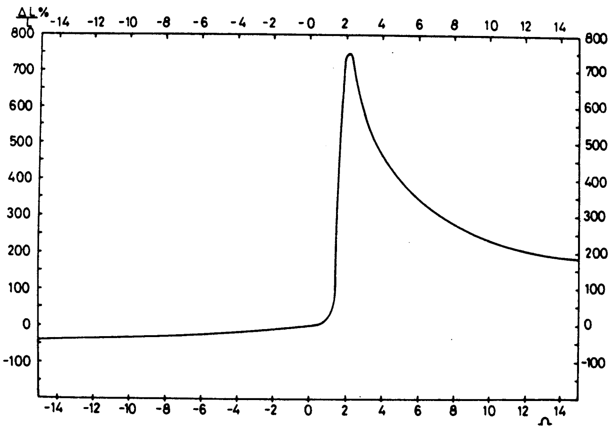

Fig. 1. Relative deviation of the luminosity from that predicted by General Relativity in percentage, $\Delta L/L$ (see (13) in the text), as a function of Ω. At $\Omega = 2.2$ there is a maximum of seven times the General Relativity value.

2. The horizon r_h wich makes $B(r_h) = 0$ fulfills the condition $r_h < r_0$ for every Ω.

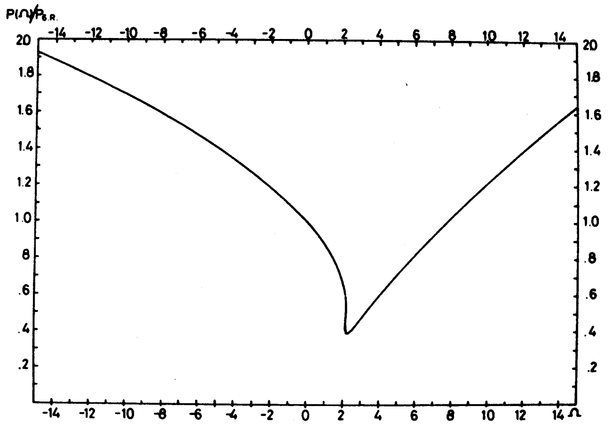

Fig. 2. Minimum orbital period of a circular orbit as a function of the parameter Ω. At $\Omega = 2.2$ exists a minimum which is almost a third of the General Relativity value.

Owing to the fact that $\nu = 0$ before r reaches r_0 (for $\Omega > 2.2$) the energy at r_z must be taken such that $\nu(r_z) = 0$. From r_z we will have spherical accretion and its contribution to total luminosity will be negligible (Novikov and Thorne 1973).

For $\Omega < 2.2$ the maximum frequency is given by $\nu(r_0)$, while for $\Omega > 2.2$ by $\nu(2.5\Omega)^{1/2}$. Figure 2 displays these values.

IV. ROTATIONAL SYMMETRY

We shall try to extend the results of section II to objects with non-negligible rotation.

The most general axially simmetric stationary metric can be written as:

$$ds^2 = -B(r,\theta)dt^2 + A(r,\theta)dr^2 + 2C(r,\theta)d\phi dt + H(r,\theta)d\phi^2 + K(r,\theta)d\theta^2. \quad (14)$$

We obtain the geodesics from the variational principle $\delta S = \int (ds/d\lambda)^2 d\lambda^2 = 0$.

$\theta = \pi/2 =$ constant is a solution to the equation of motion, also from the astrophysical point of view one expects the accretion disk to be situated in the orbital plane, because of the symmetry of the problem (Novikov and Thorne 1973).[3]

Taking $\theta = \pi/2$, the remaining equations of motion have the following integrals:

$$H(r)\, d\phi/d\lambda + C(r)\, dt/d\lambda = J = \text{constant}, \quad (15)$$

3. See also Bardeen and Petterson 1975 for other alignment mechanisms.

$$-B(r)\,dt/d\lambda + C(r)\,d\phi/d\lambda = E = \text{constant}\ , \qquad (16)$$

and choosing λ properly:

$$ds^2/d\lambda^2 = -1 = -B(r)(dt/d\lambda)^2 + A(r)\,(dr/d\lambda)^2 +$$

$$+ 2C(r)\,d\phi/d\lambda\, dt/d\lambda + H(r)\,(d\phi/d\lambda)^2 \ . \qquad (17)$$

Solving (15) and (16) for $d\phi/d\lambda$ and $dt/d\lambda$, and substituting them into (17) we obtain the effective potential equation:

$$A(BH + C^2)\,(dr/d\lambda)^2 =$$

$$= HE^2 - BJ^2 - 2CEJ - BH - C^2 = -V_{eff} + E^2 \ .$$

The three equations.

$$dr/d\lambda = \partial V_{eff}/\partial r = \partial^2 V_{eff}/\partial r^2 = 0\ ,$$

can be put in the following form:

$$\begin{aligned} &\alpha E^2 + \beta EJ + \gamma J^2 + \delta = 0\ , \\ &\alpha' E^2 + \beta' EJ + \gamma' J^2 + \delta' = 0\ , \qquad (18) \\ &\alpha'' E^2 + \beta'' EJ + \gamma'' J^2 + \delta'' = 0\ , \end{aligned}$$

where $' = d/dr$ and $\alpha = H$,

$$\beta = -2C,\ \gamma = -B,\ \delta = -(BH + C^2)\ .$$

From the first two equations (18) we find E and J at every circular orbit:

$$E^2 = [-2(\gamma'\delta - \delta'\gamma)(\alpha\gamma' - \gamma\alpha') - (\gamma\beta' - \beta\gamma') \times$$

$$\times (\delta\beta' - \beta\delta') \pm (\gamma\beta' - \beta\gamma') \times$$

$$\times \{ (\delta\beta' - \beta\delta')^2 + 4(\gamma'\delta - \delta'\gamma)(\alpha\delta' - \delta\alpha') \}^{1/2}]/$$

$$[2(\alpha\gamma' - \gamma\alpha')^2 + 2(\gamma\beta' - \beta\gamma')(\alpha\beta' - \beta\alpha')] \ , \qquad (19)$$

$$J^2 = [-2(\alpha\delta' - \delta\alpha')(\alpha\gamma' - \gamma\alpha') - (\alpha'\beta - \beta'\alpha) \times$$

$$\times (\delta'\beta - \beta'\delta) \pm (\alpha\beta' - \beta\alpha') \times$$

$$\times \{ (\delta'\beta - \beta'\delta)^2 + 4(\alpha\delta' - \delta\alpha')(\gamma'\delta - \delta'\gamma) \}^{1/2}]/$$

$$[2(\alpha\gamma' - \gamma\alpha')^2 + 2(\alpha'\beta - \beta'\alpha)(\beta\gamma' - \gamma\beta')] \ . \qquad (20)$$

In both equations the + sign corresponds to corotating orbits and the − sign to counter rotating orbits.

The orbital frequency as seen from "infinity" will be:

$$\nu(r) = d\phi/dt = (BJ + CE)/(CJ - HE) \ , \qquad (21)$$

where E and J are given by (19) and (20).

To obtain the r_0 of the last stable orbit we must solve the three equations (18) simultaneously eliminating E and J:

$$\alpha''(\delta\gamma' - \gamma\delta') + \gamma''(\alpha\delta' - \delta\alpha') + \delta''(\alpha'\gamma - \gamma'\alpha) =$$

$$= \pm \{ [\beta''(\delta'\gamma - \gamma'\delta) + \gamma''(\beta'\delta - \delta'\beta) + \delta''(\beta\gamma' - \gamma\beta')] \times$$

$$\times [\beta''(\alpha'\delta - \delta'\alpha) + \alpha''(\beta\delta' - \delta\beta') + \delta''(\alpha\beta' - \beta\alpha')] \}^{1/2} . (22)$$

Equations (19), (20), (21) and (22) are the generalizations (when rotation is taken into account) of (7), (8), (10) and (9), respectively.

As an example of application we will consider Kerr's metric:

$$ds^2 = -(1 - 2mr/\rho^2)dt^2 + [(r^2 + a^2)^2 - a^2\Delta \sin^2(\theta)] \times$$

$$\times \sin^2(\theta)/\rho^2\, d\phi^2 -$$

$$-4mra\left(\sin^2(\theta)/\rho^2\right)d\phi dt + \left(\rho^2/\Delta\right)dr^2 + \rho^2 d\theta^2 \ ,$$

$$\Delta = r^2 - 2mr + a^2 \ ,\ \rho^2 = r^2 + a^2 \cos^2(\theta) \ .$$

Taking $\theta = \pi/2$, we find:

$$\alpha = r^3 + a^2 r + 2ma^2 \ ,\ \beta = 4ma \ ,\ \gamma = 2m - r \ ,$$

$$\delta = -r^3 + 2mr^2 - a^2 r + 8m^2a^2/r \ ;$$

then, equation (22) will take the form:

$$-\tilde{r}^3 + 6\tilde{r}^2 + 3\tilde{a}^2\tilde{r} - 4\tilde{a}^2 = \pm 2\tilde{a} \times$$

$$\times [4\tilde{a}^2 - 6\tilde{a}^2\tilde{r} - 12\tilde{r}^2 + 18\tilde{r}^3]^{1/2}$$

where $\tilde{r} = r_0/m$ and $\tilde{a} = a/m$.

This equation can be solved for $\tilde{r}$ (Bardeen *et al.* 1972) giving:

$$\tilde{r} = 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + Z_2)]^{1/2} \text{ , where:}$$

$$Z_1 = 1 + [1 - \tilde{a}^2]^{1/2} \{[1 + \tilde{a}]^{1/3} + [1 - \tilde{a}]^{1/3}\} \text{ and}$$

$$Z_2 = [3\tilde{a}^2 + Z_1{}^2]^{1/2} .$$

From (19) and (20) we find E and J as follows:

$$E = [\tilde{r}^2 - 2\tilde{r} \pm \tilde{a} \ \tilde{r}^{1/2}] / [\tilde{r} \times$$

$$\times (\tilde{r}^2 - 3\tilde{r} \pm 2\tilde{a} \ \tilde{r}^{1/2})^{1/2}] ,$$

$$J = \pm[\tilde{r}^{1/2}(\tilde{r}^2 \mp 2\tilde{a}\tilde{r}^{1/2} + \tilde{a}^2)] /$$

$$[\tilde{r}(\tilde{r}^2 - 3\tilde{r} \pm 2\tilde{a} \ \tilde{r}^{1/2})^{1/2}];$$

for maximum rotation $a = m$:

$$r_0 = m \ , \ E_0 = 1/\sqrt{3} \ \text{ and } \ m\nu_m = 1/2 .$$

IV. CONCLUSIONS

This paper shows how the accretion disk luminosity and the maximum orbital frequency are modified according to different metric theories of gravitation. The results suggest that in detailed accretion disk models these effects must be taken into account. Besides, in example 4) we have seen clearly that these effects depend strongly on the metric structure in the strong field regime.

On the other hand if we had a detailed accretion disk model, we could take into account the dependence on metric theories of gravitation to obtain a gravitational test in the strong field regime.

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