Rev. Mexicana Astron. Astrof., 14, 631-635 (1987)

ASPECTS OF THE PHASE-PORTRAIT OF RESONANT PROBLEMS

WAGNER SESSIN
Departamento de Astronomia
Instituto Tecnológico de Aeronáutica
Brasil

SUMÁRIO: Este artigo analisa alguns aspectos do retrato de fase de sistemas ressonantes. Para isto são usasdos os métodos de Hori e de Delaunay extendido. Por terem características distintas levam a retratos de fase diferentes para a parte não perturbada que inclue os efeitos da ressonância. Um é definido pelo sistema auxiliar de Hori e o outro pela equação de Hamilton-Jacobi associada ao método de Delaunay. O estabelecimento do retrato de fase da parte não perturbada é essencial para o estudo das ordens superiores em sistemas ressonantes.

ABSTRACT: This paper analyses some aspects of the phase-portrait of resonant systems. The method of Hori and the extended Delaunay method are used in this study. They have distinct features and lead to different phase-portraits for the undisturbed part that includes the effects of the resonance. One is defined by the Hori auxiliary system and the other by the Hamilton-Jacobi equation associated to the extended Delaunay method. The establishment of the phase-portrait of the undisturbed part is essential for the study of the higher orders in resonant systems.

Key words: DYNAMICS — PLANETS AND SATELLITES

I - INTRODUCTION

When one intends to study resonant problems through formal series method (averaging) it is essential that the phase-portrait of the undisturbed system be defined in order to include the singular points due to the resonances.

In Lie series methods (Hori, 1966; Deprit, 1969) the phase-portrait is defined when the intermediary Hamiltonian is chosen in the form of leading terms of an expansion about the exact commensurability. We remind that it is essential that the intermediary system be completely integrable (if not, the methods are impraticable). Also in these methods, the phase-portrait defined by the intermediary Hamiltonian must be as close as possible to the phase-portrait of the actual dynamical system. Similarly to what occurs in the disturbed two-body problem without resonance. Next orders approximation may not alter substantially the phase-portrait defined by the undisturbed system. Here only internal resonances may alter this phase-portrait. However we will assume that no internal resonances occur. A bad choice of intermediate system introduces difficulties in the calculations of higher order perturbations, allowing frequencies become equal to zero (see: Ferraz-Mello and Sessin, 1984). There is not any fixed rule to chose the intermediary system. It must be completely integrable and its phase-portrait must be as close as possible to the phase-portrait of the actual dynamical system.

The extended method of Delaunay because of using the Jacobian form canonical transformation is somewhat different. At variance with Lie series method, the equations of the characteristics associated with the non-linear partial equation that defines the leading term of the transformation are not equivalent to the Hori auxiliary system. The phase-portraits defined by these two systems of ordinary differential equations are different because these two methods have distinct features. While in Hori's method secular and long-terms are treated in the same foot, in the extended Delaunay method they are treated as parameters in the equations of the characteristics.This is the fundamental difference between these two methods. The dynamical system defined by the Hori auxiliary system has the same number of degrees of freedom than the actual dynamical system. Therefore, it defines, in the region under consideration and at the required precision , the first approximation of the motion of the actual dynamical system . Higher order terms will only disturb the phase-portrait defined by the Hori auxiliary system if no internal resonances occur (as supposed here). On the other hand, the dynamical system defined by the equations of the characteristics does not have the same number of degrees of freedom than the actual dynamical system (secular and long-period terms are treated as parameters of the transformation). Therefore, it does not define a first approximation for the motion of the actual dynamical system. Higher order terms may alter substantially the phase-portrait defined by the equations of the characteristics. In this way, we say that the Hori auxiliary system freeze the phase-portrait and the extended Delaunay method freeze only partially. It is clear that the freezing of the phase-portrait is caused by the assumption that the Hori auxiliary system defines a phase-portrait as close as possible to the phase-portrait of the actual dynamical system. While in the extended Delaunay method such assumption is not made. The common assumption for both methods is that Hori auxiliary system and the equations of the characteristics must be solvable otherwise both methods are impraticable. For dynamical systems with only one degree of freedom both methods define the same phase-portrait.

II - THE RESONANT DYNAMICAL SYSTEM

Consider the dynamical system

$$\dot{x}_i = \frac{\partial \bar{F}}{\partial \theta_i} \; , \quad \dot{\theta}_i = - \frac{\partial \bar{F}}{\partial x_i} \; , \quad (1 \leq i \leq n) \tag{1}$$

whose Hamiltonian is

$$\bar{F} = \bar{F}_o(x_1) + \sum_{j \geq 1} \varepsilon^j \bar{F}_j(x_i, \theta_i) , \tag{2}$$

where ε is the small parameter. We suppose that short-periodic terms were already eliminated by some classical averaging procedure. The system is resonant in the variable θ_1, that is, the study is made in the immediate neighbourhood of $x_1 = x_{10}$ given by

$$\frac{\partial \bar{F}_o}{\partial x_{10}} = 0 . \tag{3}$$

x_{10} corresponds to the exact resonance. We could write the equations in a more general situation where m resonant variables θ_μ exist. However the already mentioned integrability condition is not easily verified for $m > 1$ (see for instance Yokoyama, 1984; Lacaz, 1985). There is not any lost in the generality in assuming $m = 1$. Long-period variables x_ρ ($\rho = 2,\ldots, n$) are generally, in Celestial Mechanics, functions of the eccentricities and inclinations which are supposed as small quantities. In this paper, x_ρ are assumed to be of the same order as $\sqrt{\varepsilon}$.

Consider now the new set of variables

$$x_1 = x_{10} + \sqrt{\varepsilon}\, y_1 , \; x_\rho = \sqrt{\varepsilon}\; y_\rho \; , \quad \theta_1 \; , \; \theta_\rho \tag{4}$$

and the time scale transformation

$$dt^* = \sqrt{\varepsilon}\; dt \tag{5}$$

then we have a new canonical sytem given by

$$\frac{dy_i}{dt^*} = \frac{\partial F}{\partial \theta_i} \; , \quad \frac{d\theta_i}{dt^*} = - \frac{\partial F}{\partial y_i} \; , \; (1 \leq i \leq n) \tag{6}$$

where the Hamiltonian is

$$F = \frac{\bar{F}}{\varepsilon} = \frac{1}{\varepsilon}\ \bar{F}_o(x_{10} + \sqrt{\varepsilon}\ y_1) + \sum_{j\geq 1} \varepsilon^{j-1}\ \bar{F}_j(x_{10} + \sqrt{\varepsilon}\ y_1, \sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) \ , \tag{7}$$

$(2 \leq \rho \leq n)$ and the constant x_{10} being calculated from the resonance condition fixed by Equation (3). Now we develop F in Taylor series in the neighbourhood of the exact resonance $x_1 = x_{10}$ and regroup the resulting terms following their order with respect to $\sqrt{\varepsilon}$. There follows

$$F = F_o + \sqrt{\varepsilon}\ F_{1/2} + \varepsilon\ F_1 + \ldots\ldots\ldots, \tag{8}$$

where

$$F_o = \frac{1}{2} A_2^o\ y_1^2 + F_1^o\ (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho), \tag{9}$$

$$F_{1/2} = \frac{1}{3!} A_3^o\ y_1^3 + B_{11}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho), \tag{10.1}$$

$$F_1 = \frac{1}{4!} A_4^o + F_2^o\ (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) + \frac{1}{2!} B_{12}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) y_1^2 \ , \tag{10.2}$$

$$F_{3/2} = \frac{1}{5!}\ A_5^o\ y_1^5 + \frac{1}{3!} B_{13}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) y_1^3 + B_{21}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) y_1 \ , \tag{10.3}$$

$$F_2 = \frac{1}{6!} A_6^o\ y_1^6 + F_3^o\ (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) + \frac{1}{4!} B_{14}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) y_1^4 +$$

$$+ \frac{1}{2!} B_{22}^o (\sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho) y_1^2 \ , \tag{10.4}$$

and so on, where

$$A_k^o = \frac{\partial^k \bar{F}_o}{\partial x_{10}^k}\ , \quad B_{jk}^o = \frac{\partial^k \bar{F}_j}{\partial x_{10}^k}\ , \quad F_j^o = \bar{F}_j(x_{10}, \sqrt{\varepsilon}\ y_\rho, \theta_1, \theta_\rho), \quad (k \geq 1,\ j \geq 1).$$

Note that $A_1^o = \partial \bar{F}_o / \partial x_{10} = 0$ is the exact resonance condition given by Equation (3). It is worth emphasizing that the functions A_k^o, B_{jk}^o, F_j^o depend on $\sqrt{\varepsilon}$ through the variable $x_\rho = \sqrt{\varepsilon}\, y_\rho$. This may lead to a new group in the Hamiltonian. This does not correspond to a new expansion of the Hamiltonian. This group comes from the fact that the Hamiltonian depends on x_ρ through eccentricities and inclinations and when we assume that x_ρ are of the order of $\sqrt{\varepsilon}$ we are assuming that eccentricities and inclinations are of this order therefore the Hamiltonian may be regrouped if the degree of eccentricities and inclinations are taken into account (for instance see Sessin and Ferraz-Mello, 1984).

In the new canonical system given by Equation (6), F_o play the role of the undisturbed Hamiltonian. The use of any formal method depends on the complete integrability of the dynamical system defined by F_o. Therefore F_o defines a new set of intermediate orbits which includes the main effects of the resonance. The phase portrait defined by F_o is closer to the actual phase-portrait than that defined by $\bar{F}_o(x_1)$. It must be emphasized that the use of $\bar{F}_o(x_1)$ to construct formal theories lead to small divisors in the series. To avoid them we have to use the new undisturbed system, defined by F_o, that contains the main effects of the resonance. The complete integrability of this new undisturbed system depends esentially on the form of F_1^o once F_o is quadratic in y_1.

As already stressed, F_1^o depends on $\sqrt{\varepsilon}$ through its arguments and we may regroup it in terms of $\sqrt{\varepsilon}$. However, the phase-portrait of the undisturbed system is defined by F_o and this fact only could be used if the resulting phase-portrait is satisfactory (topologically equivalent to the phase-portrait of the dynamical system defined by F in the region under study and at the required precisaion). The evaluation of how satisfactory is the representation of the phase-portrait is to be made by comparison of the phase-portraits obtained through numerical case studies of F.

III . THE METHOD OF HORI (1966)

Let us consider the dynamical system defined by the Hamiltonian (8). In order to apply the method of Hori we have to define an auxiliary system that is given by the undisturbed Hamiltonian F_o (Equation (9)). A general solution of the set of ordinary differential equations

$$\frac{dy_i}{d\tau} = \frac{\partial F_o}{\partial \theta_i} \ , \quad \frac{d\theta_i}{d\tau} = - \frac{\partial F_o}{\partial y_i} \ , \quad (1 \leq i \leq n) \tag{11}$$

must be obtained. If this general solution is archieved then Hori's method may be applied straightfoward. Formal solutions of the dynamical system defined by Hamiltonian (8) could be constructed. However, this dynamical system has n degree of freedom in the new frequencies of the motion that will appear may be commensurables (internal resonance). This is a consequence of the fact that Hori's method treats all variables in the same foot. In this case, the new resonant variables must be kept in the new Hamiltonian as is usual in Celestial Mechanics. If we assume that internal resonances does not occur, the phase-portrait of the resonant dynamical system (8) is frozen and given by the set of general solutions of the auxiliary system (11). The next order perturbations does not alter substantially this phase-portrait (similar to any disturbed dynamical system without resonance). It happens that, in general, the auxiliary system (11) is not integrable. We then use the smallness of the variables $x_\rho = \sqrt{\varepsilon} \ \ y_\rho$ to regroup the Hamiltonian (8) in such a way to have an intermediate dynamical system defined by

$$\tilde{F}_o(y_1, y_\rho, \theta_1, \theta_\rho) \ , \tag{12}$$

completely integrable as necessary for the application of the method. However, this choice fixes another phase-portrait that could be not topologically equivalent to the phase-portrait defined by F_o (that is topologically equivalent to the phase-portrait defined by F in the neighbourhood of the resonance). The difficulty that remains is the choice of $\tilde{F}_o$ that generates a completely integrable auxiliary system having a satisfactory frozen phase-portrait This freezing in the topological characteristics will respond for the fact that other characteristics of the dynamical system (6) may not be revealed, except those included in $\tilde{F}_o$. F_o and $\tilde{F}_o$ differ by the fact that F_o gives the exact phase-portrait in a neighbourhood of the resonance and $\tilde{F}_o$ gives an approximation of the actual phase-portrait. If the approximation is good Hori's method works well, in contrary it is impraticable.

IV - THE EXTENDED DELAUNAY METHOD (Poincaré,1893; Ferraz-Mello, 1978; Sessin, 1985)

Since Hori's method uses Lie series in order to generate the canonical transformation, the generator of these series is expressed only in terms of one set of canonical variables. However, the extended Delaunay method uses Jacobian canonical transformations whose generating function (as it is classically known) mixes old and new variables in pairs.

Let us consider the Hamiltonian F given by Equation (8) and define the canonical transformation

$$y_1 = \frac{\partial S}{\partial \theta_1}, \quad \gamma_1 = \frac{\partial S}{\partial C_1}$$

$$y_\rho = y^*_\rho + \sqrt{\varepsilon}\,\frac{\partial S}{\partial \theta_\rho}, \quad \theta^*_\rho = \theta_\rho + \sqrt{\varepsilon}\,\frac{\partial S}{\partial y^*_\rho}, \quad (\rho = 2, \ldots, n) \tag{13}$$

where $S = S(C_1, y^*_\rho, \theta_1, \theta_\rho)$. The next operations are to introduce the y_i as defined by (13) into the equation of invariance of the Hamiltonian to the transformation

$$F = F^*,$$

and the expansion of the F_j in the neighbourhood of y^*_ρ. The leading term gives

$$\frac{1}{2} A^o_2 \left(\frac{\partial S_{1/2}}{\partial \theta_1}\right)^2 + F^o_1(\sqrt{\varepsilon}\, y^*_\rho, \theta_\rho, \theta_1) = F^*_1, \tag{14}$$

where $F^*_1 = C_1$ is a constant. In this equation y^*_ρ, θ_ρ play the role of parameters since this partial differential equation does not include derivation of S with respect to θ_ρ. Once more the method can not be continued unless we may find a complete integral of Equation (14). In the case studied here it reduces simply to a quadrature and only algebraic difficulties may arise. In the case of m resonant variables θ_μ to obtain a complete integral of Equation (14) is not an easy matter as was discussed above. Here it is completely integrable and the next orders approximation may be carried out (Sessin, 1986).

At variance with the method of Hori, the solution of Equation (14) does not imply in a complete freezing of the phase-portrait of the dynamical system since the long-period terms are treated as parameters in this equation. It is worth to mention that Hori auxiliary system is not the system of the characteristics equations associated to Equation (14) since long-period terms have different treatment is each method.

In the extended Delaunay method the undisturbed system is given by a partial differential equation that could be easier to solve than the Hori auxiliary system. Here it is solvable by a simple quadrature in opposition to Hori auxiliary system that is given by 2n ordinary differential equations. Once long-period variables are treated as parameters, the extended Delaunay method depends strongly on the higher orders of approximation. They will define the motion of these variables which are defined in Hori auxiliary system when Hori's method is used. In Hori's method the phase-portrait is completely frozen by the Hori auxiliary system whose solutions may give a good description of the dynamical system. The extended Delaunay method freeze only partially the phase-portrait and may allow a much deep insight in the study of the evolution of the dynamical system. However both methods have peculiar characteristics that could be used depending on the kind of study that is made. A deep comparison of the two methods only could be made if they are applied to a problem, with the same hypotesis and initial conditions, carrying out the calculation to higher orders.

REFERENCES

Deprit, A. 1969, *Celest. Mech.*, 1, 12.
Ferraz-Mello, S. 1978, *C.R. Acad. Sc. Paris*, 286, 969.
Ferraz-Mello, S. and Sessin, W. 1984, *Celest. Mech.*, 34, 453.
Hori, G. 1966, *Publ. Astron. Soc. Japan*, 18, 287.
Lacaz, M.H.C.F. 1985, *Celest. Mech.*, 35, 9.
Poincaré, H. 1983, *Les Méthodes Nouvelles de la Mécanique Celeste*, Vol. II, (Paris: Gauthier-Villars).
Sessin, W. 1986, *Celest. Mech.*, accepted.
Sessin, W. and Ferraz-Mello, S. 1984, *Celest. Mech.*, 32, 307.
Yokoyama, T. 1984, *Celest. Mech.*, 33, 99.

Wagner Sessin: Departamento de Astronomia, Instituto Tecnológico de Aeronáutica, 12225, São José dos Campos, SP, Brasil.