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SOLUTIONS OF THE FOKKER-PLANCK EQUATION FOR THE ENERGY DISTRIBUTION OF SUPRATHERMAL ELECTRONS

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RESUMEN. Se resuelve la ecuación de Fokker-Planck (ecuación de continuidad en el espacio de energía) bajo diferentes suposiciones en relación al fenómeno de aceleración de electrones cósmicos. Las soluciones parciales obtenidas permiten la determinación de su correspondiente espectro de energía. Tres diferentes procesos de aceleración son analizados dentro del marco de la llamada "geometría-delgada" : el tiempo de aceleración característico es pequeño en comparación al tiempo de interacción colisional, tal que las pérdidas de energía durante la generación son despreciables. Se delimitan las condiciones para las cuales el enfoque-edad-energía constituye una aproximación satisfactoria a las soluciones estacionarias y no estacionarias de la ecuación de continuidad.

ABSTRACT. We solve the Fokker-Planck equation (continuity equation in the energy space) under several different assumptions, concerning the phenomena associated with acceleration of electrons. The obtained partial solutions allow for the determination of the particle energy spectra. Three different acceleration processes are worked out within the frame of a thin-geometry: the characteristic acceleration time is shorter than the collisional interaction time, so that energy losses during acceleration are negligible. We delineate the conditions for which the age-energy-approximation, approaches the stationary and non-stationary solutions of the continuity equations.

Key words: COSMIC RAYS — PARTICLE ACCELERATION

INTRODUCTION

The study of the energetic distribution of non-thermal particles is a fundamental problem in Cosmic Ray Astrophysics. The particle energy spectrum contains the information about particle generation processes, the source location and physical conditions therein. To determine particle spectra at the level of their sources several methods have been worked out; by demodulation of the observational data back to the source, taking into account the several processes that may take place during the interplanetary and interstellar propagation, or alternatively, by inferring the particle source spectrum from the deconvolution of the non-thermal electromagnetic emissions produced by the interaction of the accelerated particles, with the local matter and electromagnetic fields.

Both mentioned methods lead to a source spectrum that may be fitted by an exponential or inverse power law in energy, which by itself does not contain great information about the source phenomenology and physical conditions, but this must be inferred from additional theoretical

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work. For this later goal, usually two different approaches have been worked out in the literature; the first one consists in developing an acceleration mechanism for the particles to gain energy in the proposed electromagnetic field configuration, and deriving the corresponding energy distribution predicted by the mechanism. A more general method consists in solving a Fokker-Planck type equation of continuity in the energy space, including any kind of process during the acceleration, such as energy losses, particle transmutations and particles lost from the source volume:

To derive energy spectra, this method may entail a high degree of mathematical complexity as one considers more and more processes in the study of the phenomenon. An approximation to the stationary solution, named the age-energy approach, by analogy with the laws of radioactive decay was suggested by Fermi (1949). This approximation becomes a useful tool to search for an analytical description of the particle energy spectrum, with the consideration of several energy change processes. However at present the degree of accuracy of this approximation with respect to the stationary solution and how much information is lost with respect to the non stationary solution has not been established. So, in this paper we delimitate the ranges of particle energies and the domain of source parameters, for different acceleration mechanism, for which such an approximation is justified, as well as for the confrontation between the stationary and the non-stationary solutions.

I. The mathematical formalisms for the energy spectrum.

1. The continuity equation formalism in the energy space (Fokker-Planck-type equation). In this formalism the global evolution of particles in the energy space is described by the following equation (eg. Ginzburg and Syrovatskii, 1964)

$$\frac{\partial Ni}{\partial t} + \nabla_E(D_i \nabla N_i) + \frac{\partial}{\partial E}\left\{\left(\frac{dE}{dt}\right)_i Ni\right\} - \frac{1}{2}\frac{\partial^2}{\partial E^2}\{di\ Ni\} - q_i(E,r,t) +$$

$$P_i N_i - \sum_{k>i} \int P_{ik}(E',E)\, N_K(E,r,t)\, dE' = 0 \qquad (1)$$

Where $N_i = N_i(E,r,t)$ represents the number of particles of the kind i per total energy interval, at the position r and at time t. The 2^{nd} term represents diffusion in the energy space with diffusion coefficient D_i, the 3^{rd} term is the average systematic energy change (gain and losses) the 4^{th} term is the fluctuation in particle energy, the 5^{th} term represents injection into the source volume, the 6^{th} term is the escape of particles from the source at the rate P_i and the last term represents catastrophic disappearance or appearance of particles of the kind i, as for instance, by nuclear transmutations. The solution of the entire Equation (1) is almost impossible analytically, however, for practical purposes the complete solution is not necessary. In fact with the exception of the 1st, 3^{rd} and 5^{th} terms, the relative "weights" of the other terms are highly variable depending strongly on the species of the particle, the energy range involved, the magnetic configuration and physical conditions prevailing at the source, for instance, while for some nuclear species in galactic cosmic sources the last term may become very important, for other species such as electrons in other circumstances (e.g. solar sources) this term may not be relevant. Therefore, for the sake of simplicity let us assume that the 2^{nd}, 4^{th} and last terms do not contribute significantly to the behavior of particle evolution. Furthermore assuming spatial homogeneity at the source, Eq. (1) may be rewritten as

$$\frac{\partial N(E,t)}{\partial t} + \frac{\partial}{\partial E}\left\{\left(\frac{dE}{dt}\right)N(E,t)\right\} + \frac{N(E,t)}{\tau} = q(E,t) \qquad (2)$$

where we have setted the escape probability proportional to the mean escape time, (mean confinement time).

(a) The non-stationary solution: by applying the Laplace transformation technique, the solution of (2) is

$$N(E,t) = \frac{e^{-t^*/\tau}}{(dE/dt)} \int_{E_o}^{E} \exp(t^*/\tau)\, q(E',t)\, dE' + \frac{(dE''/dt)}{(dE/dt)} N(E'',t)\, e^{-t/\tau} \qquad (3)$$

with $t^* = \int_{E_{th}}^{E} \frac{dE'}{(dE/dt)}$, where we have considered $E_o = E_{th} = 3/2(kT)$

with k=Boltzman constant and T (°K)=the temperature in the source, E" is obtained from the condition $t - \int_{E''}^{E} dE'/(dE'/dt) = 0$. In Eq. (3), q (E',t) is the so called "source term", i.e. the spectrum of the flux particle injected into the acceleration process.

(b) The stationary solution: we consider the case when the generation process has reached a stationary state, so, Eq.(2) losses its temporal dependence, in which case the solution is

$$N(E) = \frac{e^{-t^*/\tau}}{(dE/dt)} \int_{Eo}^{E} q(E') e^{t^*/\tau} dE' + \frac{N(E_{th}) e^{-t^*/\tau} (dE/dt)_{Eth}}{(dE/dt)} \quad (4)$$

where $N(E_{th})$ and $(dE/dt)_{E_{th}}$ are the energy distribution and the rate of energy change evaluated at E_{th} respectively.

2. The age-energy formalism. According to Fermi (1949) an analogy between particle behavior in the energy space and radioactive decay in the time space may be established, such that

$$N(E)dE = N(t)dt = (No/\tau) \times \exp(-t^*/\tau)dt \quad (5)$$

where No is the initial number of particles participating in the process, τ is the mean confinement time which characterize the remaining time of particles within the process(in the source volume), t^* is the duration of the process and is derived from the net energy change rate(dE/dt).

II. The acceleration rates:

We have chosen three different acceleration processes, with the aim of generalization of our analysis.

1) Impulsive acceleration by deterministic electric fields generated, for instance, by magnetic reconnection in a magnetic neutral sheet. Hereafter we will denominate this kind of electric field acceleration by neutral layer (NL) acceleration, whose well know energy gain rate is

$$(dE/dt)_{NL} = K_1 \beta \quad (6)$$

where $\beta = v/c$ is the particle velocity in terms of the light velocity and K_1 is the acceleration efficiency which depends on the average electric intensity in the acceleration region($\mathcal{E}$).

2) The Betatron mechanism, whose well known acceleration rate is

$$(dE/dt)_B = \alpha_B \beta^2 E \quad (7)$$

where the acceleration efficiency α_B depends on the temporal gradient of the magnetic field.

3) The Fermi-Mechanism, which process is associated with the spatial variations of the magnetic field, and the energy rate, is of stochastic nature.

$$\left(\frac{dE}{dt}\right)_f = \alpha_f \beta E \quad (8)$$

where the acceleration efficiency α_f depends on the ratio of the accelerating magnetic scatter centers and the velocity of particles.

III. The astrophysical scenarios:

Scenario # 1

Here we consider the case where the only particles that participate in the acceleration process are local particles, whose initial energy distribution is of the Maxwelian-type, that is, there is no "source term", which implies that the injection spectrum $q(E,t)=0$. This scenario corresponds to a first acceleration phase.

Scenario # 2.

Here we consider the case in which the acceleration process is of selective nature with respect to particle energy, only particles with energy above a certain threshold value $E_i >> (3/2)kT$ participate. We assume an injection spectrum from neutral sheet acceleration, as obtained from the stationary formalism within the frame of Scenario 1. This scenario corresponds to a secondary phase of acceleration.

Scenario # 3.

This includes the two first scenarios, as the more general case. Here there is no threshold energy value for acceleration and we are dealing with a secondary acceleration phase, since the acceleration region is nourished with previously accelerated particles by an impulsive mechanism similar to scenario # 2. Therefore, both populations, local thermal and pre-accelerated particles, participate in the acceleration process.

IV. Energy Spectra of the Accelerated Electrons

1. First Scenario

a) Age-energy formalism

i) For the Fermi mechanism it is obtained

$$N(E)=\frac{AN_o'}{\alpha_f\tau}\frac{\{E+(E^2-m^2c^4)^{0.5}\}^{-1/\alpha_f\tau}}{(E^2-m^2c^4)^{0.5}} \quad \text{with } A=\{E_{th}+(E_{th}^2-m^2c^4)^{0.5}\}^{\frac{1}{\alpha_f\tau}} \; ; \; E_o=1.5KT \qquad (9)$$

ii) For Betatron acceleration we have

$$N(E)=\frac{BN_o'}{\alpha_B\tau}\{E^2-m^2c^4\}^{-1-(1/2\alpha_B\tau)} \quad \text{with } B=\{E_{th}^2-m^2c^4\}^{\frac{1}{2\alpha_B\tau}} \qquad (10)$$

iii) For neutral sheet acceleration we have

$$N(E)=\frac{CN_o'E}{\tau k_1}\frac{\exp\{-(E^2-m^2c^4)^{0.5}/\tau k_1\}}{(E^2-m^2c^4)^{0.5}} \quad \text{with } C=\exp\{(E_{th}^2-m^2c^4)^{0.5}/\tau k_1\} \qquad (11)$$

with $k_1= 2.895\times10^{10}\,\varepsilon$ and ε= electric field. Under this formalism N_o' =nV is the total number of electrons in the source, with n=electrons/cm^3 and V in cm^3 is the volume of the acceleration region. For the particular case of the Fermi mechanism we are considering that only particles with velocities for above the local Alfven velocity are accelerated such that subtracting the integral spectrum of the Maxwellian distribution we have

$$N_o=nV\{1-1.4105\times10^6T^{-3/2}\int_0^{E_a}E_k^{0.5}\exp(-E_k/kT)\,dE_k\} \text{ (electrons)} \qquad (12)$$

Where E_k is the kinetic energy and E_a the corresponding Alfven energy.

b) The stationary-solution formalism

Since we are within the first scenario (q=0) the first term of Eq.(4) disappears and the energy spectra becomes similar to those given in Eqs. (9)-(11) for the age-energy formalism, multiplied by the differential energy spectrum of the Maxwellian distribution (electrons/eV), evaluated at E_k=1.5kT, and the corresponding acceleration rates evaluated at this same characteristic thermal energy value.

c) The non-stationary solution formalism

Here again the first term of Eq. (3) disappears and the energy spectra becomes

i) For the Fermi-mechanism

$$N(E,t)=N_1E_k'' \exp(-t/\tau-E_k''/kT)\ (E''^2-m^2c^4)^{0.5} / (E^2-m^2c^4)^{0.5} \quad (13)$$

where $E''=Ee^{-\alpha_f t}$

ii) For the betatron acceleration

$$N(E,t)=N_1E_k'' \exp(-t/\tau-E_k''/kT)E(E''^2-m^2c^4)/E''(E^2-m^2c^4) \quad (14)$$

with $E''=\{(mc^2)+(E^2-m^2c^4)\ e^{-2\alpha_B t}\}^{0.5}$

iii) For the neutral sheet acceleration process (NL)

$$N(E,t)=N_1E_k'' \exp(-t/\tau-E_k''/kT)E(E''^2-m^2c^4)^{1/2}/E''(E-m^2c^4)^{1/2} \quad (15)$$

with $E''=\{E^2+(k_1t)^2-2(k_1t)(E^2-m^2c^4)^{1/2}\}^{0.5}$

In Eq. (13)-(15) $N_1=2\pi nV/(\pi kT)^{3/2}$ is the coefficient of the differential thermal spectrum.

2. 2nd Scenario

Here as we are dealing with a 2nd phase acceleration, only the Fermi process was analyzed, through the evaluation of t*.

(a) The age-energy formalism:

In this case, as the injection spectrum does not appear explicitely, in order to make comparisons with the other formalisms, we considered the injection spectrum as the No value appearing in Eq. (5). Therefore, for No we took the energy spectrum which appears in the precedent stationary-solution formalism within the first scenario, with neutral sheet acceleration.

(b) The stationary-solution formalism:

Here the 2nd term in Eq. (4) disappears and the q(E') spectrum is again as in the previous case, the stationary solution with neutral sheet acceleration described in the 1st Scenario.

(c) The non-stationary-solution formalism:

Here the 2nd term of Eq.(4) disappears and for q(E',t) we employed Eq. (11).

3. 3rd Scenario

Since this scenario corresponds also to a 2nd phase acceleration, only the Fermi process was considered for the evaluation of t*.

(a) The age-energy formalism:

Here we have considered within No the addition of the thermal population and the integral spectrum of the stationary solution with neutral sheet acceleration.

(b) The stationary solution formalism

The 1st and 2nd terms of Eq. (4) corresponds to the stationary solutions of Scenarios 1 and 2 respectively.

(c) The non-stationary solution formalism.

The 1st and 2nd terms of Eq. (3) correspond to the non-stationary solutions of Scenarios 1 and 2 respectively.

V. Results

For comparisons we have proceed to evaluate ratios between the two formalisms within the frame of the three scenarios, for a wide range of parameters prevailing in astrophysical cosmic rays sources, such as displayed in Table 1.

Some of the intercomparisons between the three formalisms has been displayed in Fig. 1-8. The parameters of the source and injection regions, used in Fig. 2-8 have been tabulated in Table 2.

Since the comparison is made by means of a ratio, it becomes independent of density and volume. The temperature was varied between 10^5 and 10^8 °K.

Table 1

Source Parameters		Range
Acceleration efficiency ,	$\alpha(s^{-1})$	$0.01 < \alpha < 10$
Electric field,	ε (volt/cm)	$10^{-4} < \varepsilon < 10^{-7}$
Density,	n(elec./cm³)	$10^8 \le n \le 10^{14}$
Temperature,	T (°K)	$5\times10^5 \le T \le 10^8$
Magnetic Field,	B (gauss)	$10 \le B \le 100$
Source volume,	V (cm³)	$10^{20} \le V \le 10^{38}$
Confinement time,	τ (s)	$10^{-2} \le \tau \le 5$
Evolution time,	t (s)	$2 < t < 5$

VI. Analysis of the results

Within the frame of the 1st Scenario, it can be seen in Fig. 1 that the agreement between the age-energy formalism and the stationary formalism is excellent, and independent of particle energy: there is no real difference when the betatron process is considered, and for the two other acceleration mechanisms the ratio falls between 0.4 and 2 for temperatures of the order of 10^5-10^6 °K. On the other hand the ratio of the non-stationary formalism and the age-energy

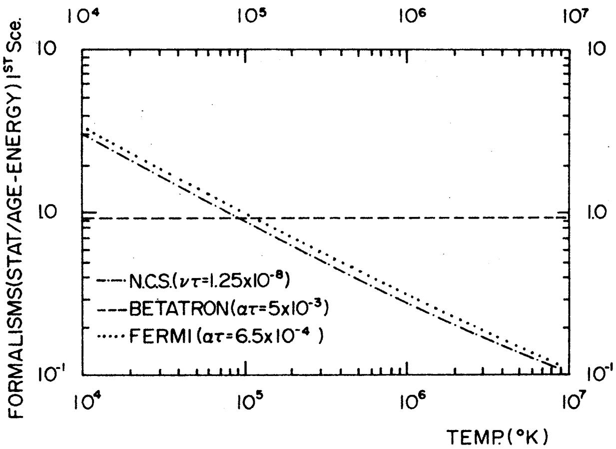

Fig. 1. Intercomparisons between the three formalisms.

formalism (Fig.2-4) show that they are not comparable, since the best agreements occur only for a very short energy range and very high temperature. Obviously the same result is obtained from the ratio of the non-stationary and stationary formalisms.

Within the frame of Scenario # 2, it can be seen from Fig. 5 that the agreement between the stationary and the age-energy formalisms is quite satisfactory between 20 keV and 10 MeV and temperatures in the range (5×10^5-10^7) °K, and only at high energies there may appear slight differences. For the intercomparison with the non-

TABLE 2

FIG.	REGION OF ACELERATION	REGION OF INJECTION
2	$\varepsilon = 10^{-6}$, τ= 1., n = 10^{10}, t = 5, V=1	-
3	α= 0.25, τ=4, n=10^{10}, B=35, t =4	-
4	$0.1 \le \alpha\tau < 1$, n=10^{10} , B=10, t =8	-
5	α= 0.25, τ=4, n=10^{10}, B=10	ε= 2×10^{-7} , τ=1, n=10^{11}
6	$0.1 \le \alpha\,\tau < 1$, n=10^{10} , t =8, E_0=15KeV	ε= 10^{-6} , τ=1, n=10^{11}
7	$0.1 < \alpha\tau < 1$, n=10^{10}, B=10,t =8	ε=10^{-7} , τ=1, n=10^{11}
8	$0.1 < \alpha\,\tau < 1$, n=10^{10}, B=10, t=8 E=15KeV	ε=1.5×10^{-7} , τ=1, n=10^{11}

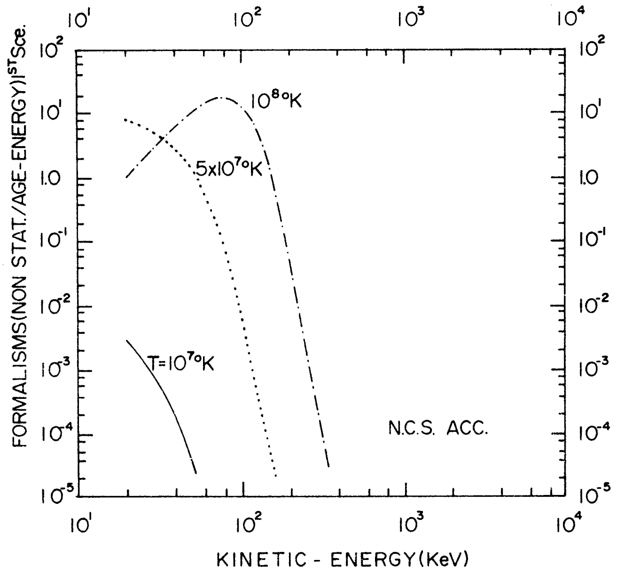

Fig. 2. Same as Figure 1.

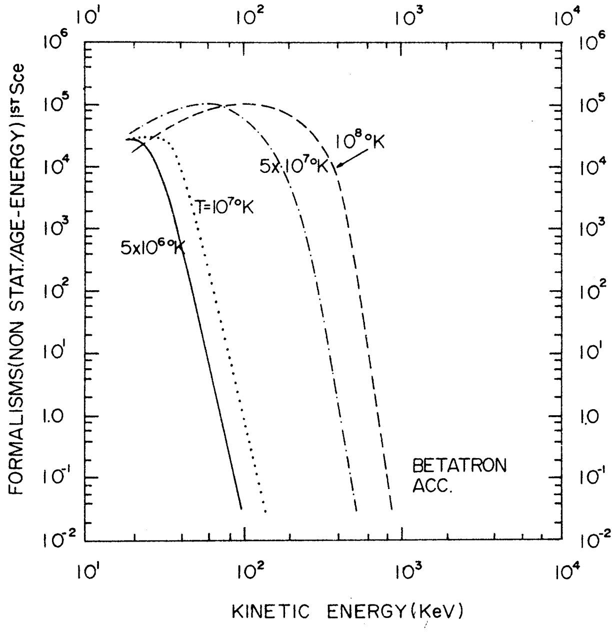

Fig. 3. Same as Figure 1.

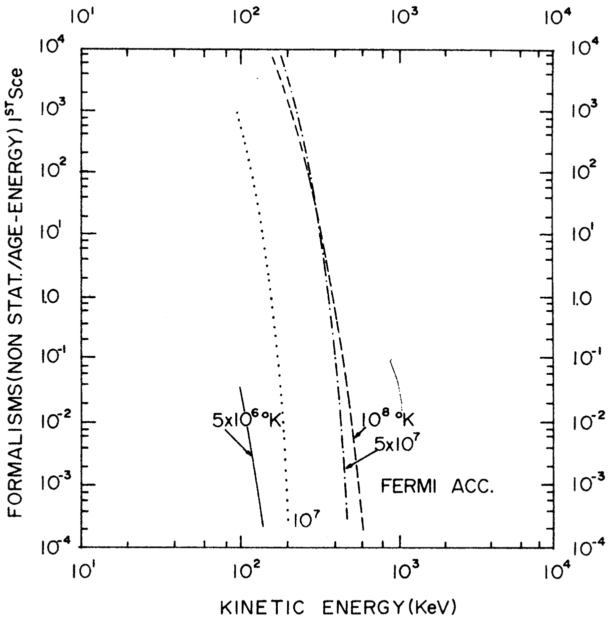

Fig. 4. Same as Figure 1.

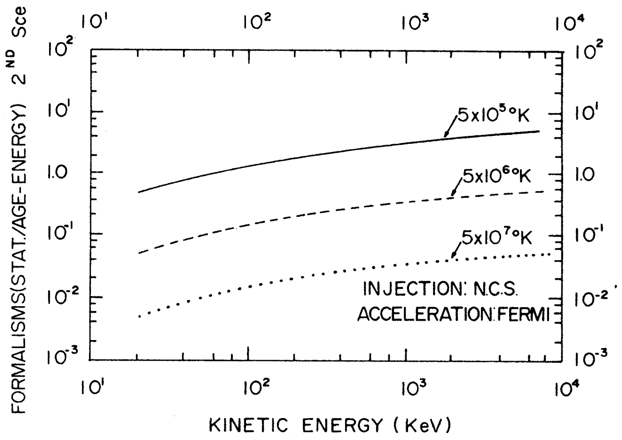

Fig. 5. Same as Figure 1.

stationary formalism, the only agreement may appear in the short energy range (20-300) KeV when the source temperature is higher than 10^6 °K., as it is illustrated in Fig. 6 and 7.

For Scenario # 3 the stationary and age-energy formalisms are more similar between 20 KeV and 10 MeV for T=10^6 °K. For the intercomparisons with the non-stationary formalisms, the agreement is again very poor, becoming reasonable only for 1-3 MeV and T >10^6 °K.

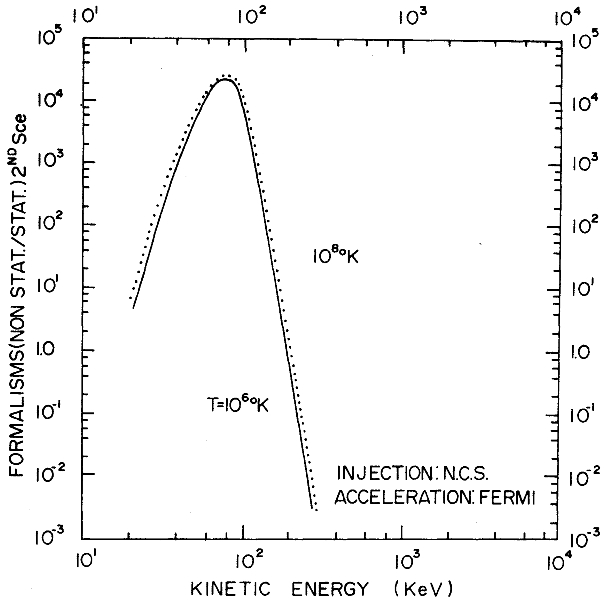

Fig. 6. Same as Figure 1.

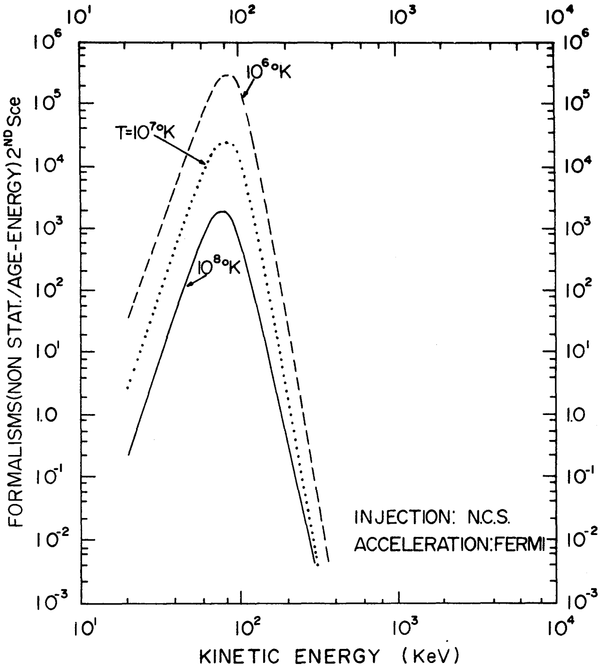

Fig. 7. Same as Figure 1.

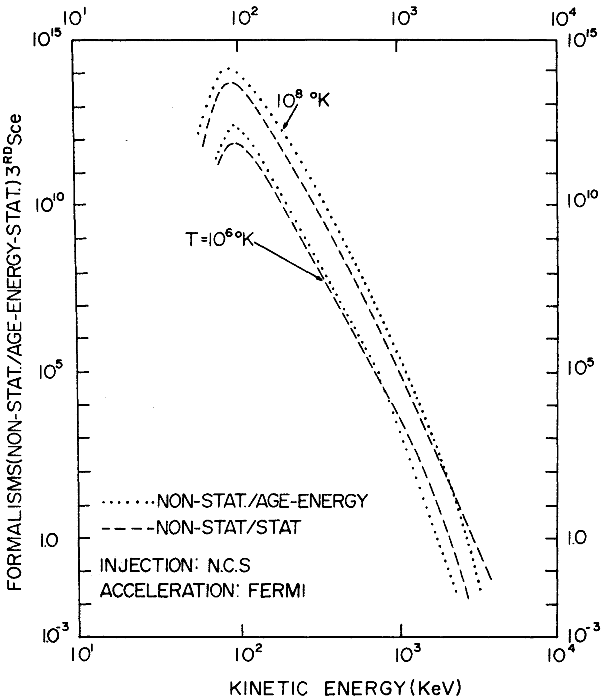

Fig. 8. Same as Figure 1.

VII. Conclusions

From the above analysis it is clear that both the age-energy, and stationary formalisms are very similar and in general they may be used indistinctly when the mathematical complexity of a given problem may be simplified by the substitution of one by the another. As can be seen from Eqs. (4) and (5) it is expected that a given problem with the age-energy approach will be solvable easily. On the other hand, it is also clear that neither the stationary nor the age-energy formalism can substitute the non-stationary approach with the exception a specific short energy range and very high source temperatures. The non-stationary solution would eventually tend towards the stationary one for evolution times (t) much longer than that employed in this work. Therefore for very short times (for instance at the beginning of a solar particle event) the stationary or age-energy formalisms are not valid.

References

Fermi, E. 1949, Phys. Rev. 75, 1169.
Ginzburg V.L. and Syrovatskii, S'I. 1964. "The origin of Cosmic Rays" ed. by Pergomon Press.
Miroshnichenko, L.I. and Petrov, B.M., 1985, "Dynamics of Radiation Conditions in Space", ed. by Energoatmizdat, Moscow.
Pérez-Peraza, J., 1986 Space Sc. Rev. 44, 91-138.

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