Rev. Mexicana Astron. Astrof., 20, 15-22 (1990)

ON EQUILIBRIUM FIGURES FOR IDEAL FLUIDS IN THE FORM OF CONFOCAL ELLIPSOIDS ROTATING WITH COMMON ANGULAR VELOCITY

F.J. Martínez, J. Cisneros, and D. Montalvo

Universidad Autónoma de San Luis Potosí, México

Received 1989 July 11

RESUMEN

Se demuestra la existencia de figuras de equilibrio para un cuerpo fluído, autogravitante y libre de presión externa, que consiste de dos elipsoides *confocales* de distinta densidad que giran con velocidad angular común. El análisis muestra que a cada valor asignado a la densidad relativa del cuerpo, le corresponde un solo grado de achatamiento, es decir, no puede existir una *serie*. Otra conclusión es que la densidad relativa posee un límite inferior (tal que la densidad del elipsoide interior es ligeramente mayor que el doble de la del exterior) y al respecto se ofrece una explicación semicualitativa. Se asume que el fluído es ideal e incompresible.

ABSTRACT

For a self-gravitating and free from external pressure fluid body, consisting of two homogeneous *confocal* ellipsoids of different density rotating with common angular velocity, we demonstrate the existence of equilibrium figures. No *series* is possible, however, since the rotating body attains, for given values of its relative density, a unique degree of flattening. In addition, the analysis shows that there is a lower limit to the relative density (in which case the density of the interior ellipsoid is only slightly larger than twice the density of the exterior one) and a semi-qualitative explanation on the subject is offered. The fluids are assumed ideal and incompressible.

Key words: **HYDRODYNAMICS**

I. INTRODUCTION

Incompressible, self-graviting homogeneous flu- ds are known to adopt the spheriodal form (fol- owing McLaurin) or the ellipsoidal one (following acobi) when rotating at constant angular velocity bout the figure's minor axis. The spheroids, which nust be oblate, can have any value of angular nomentum while the ellipsoids require for their xistence an angular momentum greater than a ertain finite value (Lyttleton 1951).

In this work, we undertake the problem of quilibrium figures for a rotating *composite* fluid ody. The specific model consists of two concentric, nequal density ellipsoids, each of which is made ut of an incompressible, homogeneous, ideal fluid, vith the additional assumption that the body as whole, is self-graviting and free from external ressure. The model may be thought of as an llipsoid of a given density surrounded by another llipsoid of lower density. We have therefore etaken, to express it in a figurative way, Jacobi's research line, inasmuch as the present model also involves ellipsoidal geometry. We study the special case in which both ellipsoids of the model assumed to be confocal share a common angular velocity, and we inquire about equilibrium figures by the suitable application of boundary conditions.

The selection of a common angular velocity, rather than the case in which each homogeneous part rotates with its own, was dictated by the wish to overcome dynamical effects; the confocality assumption, on the other hand, is adopted for the sake of mathematical economy (see next section).

II. POTENTIAL AND EQUILIBRIUM CONDITIONS

In this section, we shall show that the application of suitable boundary conditions on our model yields a set of equations, the analysis of which will determine whether or not equilibrium figures are possible

a) *Homogeneous Fluids*

Since our model requires the consideration of homogeneous fluids, we begin by writing out the potential of a homogeneous ellipsoid.

i) The Potential of a Homogeneous Ellipsoid.

The interior potential of a homogeneous ellipsoid is given by the integral relation (Chandrasekhar 1969):

$$B = \pi G \rho a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta}\left(1 - \frac{x_i^2}{a_i^2 + u}\right), \qquad (1)$$

where ρ is its density, a_1, a_2, a_3 its semiaxes, G the gravitational constant and

$$\Delta = \left[(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)\right]^{1/2}.$$

Expression (1) can indiscriminately be used for exterior points as well, if λ, the ellipsoidal coordinate of the considered point, i.e., the positive root of the cubic equation

$$\sum_i \frac{x_i^2}{a_i^2 + \lambda} = 1\,, \qquad i = 1,2,3 \qquad (2)$$

is taken as the lower limit of the integral.

ii) The Equilibrium Conditions for an Arbitrary Homogeneous Fluid.

If the fluid is at rest and under the action solely of conservative forces, the equilibrium condition is provided at once by Euler's equation particularized for zero velocity and integrated over the volume of the fluid (Milne-Thomson 1968):

$$p = \rho B + constant, \qquad (3)$$

where p is the pressure, ρ the density and B the gravitational potential. If, however, the fluid is rotating (as we will assume henceforth), then eq. (3) must be modified into

$$p/\rho = B + \frac{\omega^2}{2}(x_1^2 + x_2^2) + constant, \qquad (4)$$

where $\vec{\omega} = \omega\hat{k}$, and $\hat{k}$ is the unit vector along x_3, the rotation axis. In particular, on the surface of the fluid eq. (4) becomes

$$B + \frac{\omega^2}{2}(x_1^2 + x_2^2) + constant = 0, \qquad (5)$$

if the fluid is to be considered free from any external pressure.

Calling φ the left hand side of eq. (5) and writing the surface equation as $f(x_1, x_2, x_3) = 0$, the following ratios must be satisfied (Lyttleton 1951)

$$\varphi_{x_1}/f_{x_1} = \varphi_{x_2}/f_{x_2} = \varphi_{x_3}/f_{x_3}, \qquad (6)$$

where the x_i subscripts stand for partial derivatives

b) *The Model*

We now turn our attention to the model (we shall currently refer to its innermost ellipsoid as the "nucleus"and to its envelope as the "atmosphere"). We designated by the subscript $n(a)$ quantities pertaining the nucleus (atmosphere) and, as a first step, we derive the potentials $B_a(x_1, x_2, x_3)$ and $B_n(x_1, x_2, x_3)$, at each point of the atmosphere and nucleus, respectively. We assume $\rho_n > \rho_a$.

i) The Potentials.

The derivation of B_a is accomplished by following the sequence of steps indicated in Figure 1 of our previous work on spheroids (Montalvo, Martínez and Cisneros, 1983). We have

$$B_a = \phi G\left[\rho_a a_{a_1} a_{a_2} a_{a_3} \int_0^\infty \frac{du}{\Delta_n}\left(1 - \sum \frac{x_i^2}{a_{a_i}^2 + u}\right) - \right.$$

$$\left. + (\rho_n - \rho_a) a_{n_1} a_{n_2} a_{n_3} \int_0^\infty \frac{du}{\Delta_n}\left(1 - \sum \frac{x_i^2}{a_{n_i}^2 + u}\right)\right], \qquad (7)$$

where

$$a_{a_i}(a_{a_1} > a_{a_2} > a_{a_3})$$

and

$$a_{n_i}(a_{n_1} > a_{n_2} > a_{n_3})$$

stand for the semiaxes of the atmosphere and the nucleus, respectively; here

$$\Delta_a = \left[(a_{a_1}^2 + u)(a_{a_2}^2 + u)(a_{a_3}^2 + u)\right]^{1/2}$$

and

$$\Delta_n = \left[(a_{n_1}^2 + u)(a_{n_2}^2 + u)(a_{n_3}^2 + u)\right]^{1/2}.$$

The expression for B_n is also given by eq. (7) writing $\lambda = 0$ as the lower limit of the second integral.

ii) The Confocality Condition.

As in our previous work, we limit ourselves to treat the case in which the nucleus and the atmosphere are confocal. The hypothesis of confocality, vhich can only be fully justified *a posteriori*, is for he sake of mathematical simplicity, because in this ›articular ellipsoidal geometry, the quantity λ [see :q. (2)] is constant over the external border of the ›ody. With this in mind, we have

$$ {}^{2}_{a_1} = a^2_{n_1} + \lambda, \qquad a^2_{a_2} = a^2_{n_2} + \lambda, \qquad a^2_{a_3} = a^2_{n_3} + \lambda; $$

›r, equivalently

$$ a^2_{n_1} - a^2_{n_2} = a^2_{n_1} e^2_{n_1}, \qquad a^2_{n_1} - a^2_{n_3} = a^2_{n_1} e^2_{n_2}, $$

$$ a^2_{a_1} - a^2_{a_2} = a^2_{a_1} e^2_{a_1}, \qquad a^2_{a_1} - a^2_{a_3} = a^2_{a_1} e^2_{a_2}, $$

›here the convention is made that $e_{n_1}(e_{a_1})$ and $_{n_2}(e_{a_2})$ stand for the equatorial and the meridional ccentricities of the nucleus (atmosphere) respec- ively.

As a consequence of the above expressions, there esults:

$$ \frac{e_{n_1}}{e_{a_1}} = \frac{a_{a_1}}{a_{n_1}} > 1, \quad \text{and} \quad \frac{e_{n_2}}{e_{a_2}} = \frac{a_{a_1}}{a_{n_1}} > 1, \quad (8) $$

›hich means that the eccentrities of the nucleus are reater than those of the atmosphere.

Further, from the confocality condition it also ›llows that

$$ \frac{e_{a_1}}{e_{n_1}} = \frac{e_{a_1}}{e_{n_1}}, \quad (9) $$

› the eccentricities cannot be independent of each ther and one eccentricity, say e_{a_1}, can be dropped ut of thc problem; henceforward in our treatment, ›is eccentricity will not appear.

iii) Boundary Conditions for Rigid Body Rotation.

We now proceed to obtain expressions for the ngular velocity of the body in terms of the densities nd the eccentricities, as well as the elliptic integrals ›hich are implicit in the expressions for B_a and B_n.

To this end, we will apply the condition already ›ecified in the introduction, namely, that the body › to be considered free from external pressure; in ddition, we demand continuity of the pressure at ›e boundary surface between the nucleus and the tmosphere (Landau and Lifshitz 1959).

Displaying the above conditions in more detail ›ey read as [see eq. (4)]

$$ \varphi \equiv B_a + \frac{1}{2}\omega^2(x_1^2 + x_2^2) + constant = 0, \quad (10) $$

nd (defining φ' as the left side of $p_a - p_n = 0$)

$$ \begin{aligned} \varphi' \equiv\ & \rho_a B_a + 1/2\, \rho_a \omega^2 (x_1^2 + x_2^2) - \\ & - \rho_n B_n - 1/2\, \rho_n \omega^2 (x_1^2 + x_2^2) + constant = 0, \quad (11) \end{aligned} $$

respectively.

Consider first eq. (10). From eq. (6) and

$$ f_a = \frac{x_1^2}{a^2_{a_1}} + \frac{x_2^2}{a^2_{a_2}} + \frac{x_3^2}{a^2_{a_3}} - 1 = 0, $$

the surface equation of the atmosphere, we obtain the following two expressions:

$$ \begin{aligned} \Omega^2 = & \left[\frac{(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}(1-e^2_{a_2})}{e_{n_2}(e^2_{n_2}-e^2_{n_1})} \right. \\ & \left. - \frac{(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}}{e_{n_2}e^2_{n_1}} \right] \varepsilon E_n \\ & + \frac{(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}}{e_{n_2}e^2_{n_1}} \varepsilon F_n \\ & + \left[\frac{e_{n_2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}(1-e^2_{a_2})^{3/2}}{e^3_{a_2}(e^2_{n_2}-e^2_{n_1})} \right. \\ & \left. - \frac{e_{n_2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}}{e^2_{n_1}e^3_{a_2}} \right] E_a \\ & + \frac{e_{n_2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}(1-e^2_{a_2})^{1/2}}{e^2_{n_1}e^3_{a_2}} F_a \\ & - \frac{e_{a_2}(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}(1-e^2_{a_2})^{1/2}}{e^2_{n_2}(e^2_{n_2}-e^2_{n_1})} \varepsilon \\ & - \frac{(e^2_{n_2}-e^2_{n_1}e^2_{a_2})(1-e^2_{a_2})}{e^2_{a_2}(e^2_{n_2}-e^2_{n_1})}, \qquad (12) \end{aligned} $$

and

$$ \begin{aligned} \Omega^2 = & \left[\frac{e_{n_2}(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}(1-e^2_{a_1})}{(e^2_{n_2}-e^2_{n_1})(e^2_{n_2}-e^2_{n_1}e^2_{a_2})} \right. \\ & \left. + \frac{e_{n_2}(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}}{e^2_{n_1}(e^2_{n_2}-e^2_{n_1})} \right] \varepsilon E_n \\ & - \frac{(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}}{e_{n_2}e^2_{n_1}} \varepsilon F_n \\ & + \left[\frac{e^3_{n_2}(1-e^2_{a_2})^{3/2}}{e^3_{a_2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}(e^2_{n_2}-e^2_{n_1})} \right. \\ & \left. + \frac{e^3_{n_2}(1-e^2_{a_2})^{1/2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}}{e^2_{n_2}e^3_{a_2}(e^2_{n_2}-e_{n_1})} \right] E_a \\ & - \frac{e_{n_2}(1-e^2_{a_2})^{1/2}(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}}{e^2_{n_1}e^3_{a_2}} F_a \\ & - \frac{2e_{a_2}(1-e^2_{n_1})^{1/2}(1-e^2_{n_2})^{1/2}(1-e^2_{a_2})^{1/2}}{(e^2_{n_2}-e^2_{n_1}e^2_{a_2})^{1/2}(e^2_{n_2}-e^2_{n_1})} \varepsilon \\ & - \frac{2e^2_{n_2}(1-e^2_{a_2})}{e^2_{a_2}(e^2_{n_2}-e^2_{n_1})^{1/2}}, \qquad (13) \end{aligned} $$

where

$$\Omega^2 \equiv \frac{\omega^2}{4\phi G\rho_a}. \qquad \text{and} \qquad \epsilon = \frac{\rho_n - \rho_a}{\rho_a},$$

and

$$F_a = \int_0^{\varsigma} e_{a_2}\left(e_{a_2}^2 - \frac{e_{n_1}^2 e_{a_2}^2}{e_{n_2}^2} sin^2 w\right)^{-1/2} dw,$$

$$E_a = \int_0^{\varsigma}\left(e_{a_2}^2 - e_{n_1}^2 e_{a_2}^2 e_{n_2}^2 sin^2 w\right)^{1/2} e_{a_2}^{-1} dw,$$

are the familiar elliptic integrals of the first and second kind, respectively, with their upper limit given by $\varsigma = cos^{-1}\left(1 - e_{a_2}^2\right)^{1/2}$ (MacMillan 1958).

In an entirely similar way, and using the fact that λ vanishes at the interface nucleus-atmosphere, we obtain from equations (11), (6) and

$$f_n = \frac{x_1^2}{a_{n_1}^2} + \frac{x_2^2}{a_{n_2}^2} + \frac{x_3^2}{a_{n_3}^2} - 1 = 0,$$

the surface equation of the nucleus, the following two expressions:

$$\begin{aligned}
\Omega^2 = &\left[\frac{\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{3/2}}{e_{n_2}\left(e_{n_2}^2 - e_{n_1}^2\right)} - \frac{\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{e_{n_2} e_{n_1}^2}\right]\epsilon E_n \\
&+ \frac{\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{e_{n_2} e_{n_1}^2}\epsilon F_n \\
&+ \left[\frac{e_{n_2}\left(1 - e_{n_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}\left(1 - e_{a_2}^2\right)^{1/2}}{e_{a_2}^3\left(e_{n_2}^2 - e_{n_1}^2\right)} - \frac{e_{n_2}\left(1 - e_{a_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}}{e_{n_1}^2 e_{a_2}^3}\right] E_a \\
&+ \frac{e_{n_2}\left(1 - e_{a_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}}{e_{n_1}^2 e_{a_2}^3} F_a \\
&- \frac{\left(1 - e_{n_1}^2\right)\left(1 - e_{n_2}^2\right)}{\left(e_{n_2}^2 - e_{n_1}^2\right)}\epsilon \\
&- \frac{\left(1 - e_{n_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)}{e_{a_2}^2\left(e_{n_2}^2 - e_{n_1}^2\right)},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\Omega^2 = &\left[\frac{\left(1 - e_{n_2}\right)^{3/2}}{e_{n_2}\left(1 - e_{n_1}^2\right)^{1/2}\left(e_{n_2}^2 - e_{n_1}^2\right)} + \frac{e_{n_2}\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{e_{n_1}^2\left(e_{n_2}^2 - e_{n_1}^2\right)}\right]\epsilon E_n \\
&- \frac{\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{e_{n_2} e_{n_1}^2}\epsilon F_n \\
&+ \left[\frac{e_{n_2}\left(1 - e_{n_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}\left(1 - e_{a_2}^2\right)^{1/2}}{e_{a_2}^3\left(1 - e_{n_1}^2\right)\left(e_{n_2}^2 - e_{n_1}^2\right)} + \frac{e_{n_2}^3\left(1 - e_{a_2}^2\right)^{1/2}\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}}{e_{n_1}^2 e_{a_2}^3\left(e_{n_2}^2 - e_{n_1}^2\right)}\right] E_a \\
&+ \frac{e_{n_2}\left(1 - e_{a_2}^2\right)^{1/2}\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)^{1/2}}{e_{n_1}^2 e_{a_2}^3} F_a \\
&- \frac{2\left(1 - e_{n_2}^2\right)}{\left(e_{n_2}^2 - e_{n_1}^2\right)}\epsilon - \frac{e_{n_2}^2\left(1 - e_{a_2}^2\right)}{e_{a_2}^2\left(e_{n_2}^2 - e_{n_1}^2\right)} \\
&- \frac{\left(1 - e_{n_2}^2\right)\left(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2\right)}{e_{a_2}^2\left(e_{n_2}^2 - e_{n_1}^2\right)\left(1 - e_{n_1}^2\right)},
\end{aligned} \tag{15}$$

where

$$F_n = \int_0^{\xi}\frac{e_{n_2}}{\left(e_{n_2}^2 - e_{n_1}^2 sin^2 w\right)^{1/2}} dw,$$

$$E_n = \int_0^{\xi}\frac{\left(e_{n_2}^2 - e_{n_1}^2 sin^2 w\right)^{1/2}}{e_{n_2}} dw,$$

are the corresponding elliptic integrals, with the upper limit given by $\xi = cos^{-1}\left(1 - e_{n_2}^2\right)^{1/2}$.

Before going on any further, we may notice, by the way, that the well known results on equilibrium figures for rotating homogeneous ellipsoids (Jacobi's figures) are self-contained in our equations: If we let $e_{n_1} \to e_{a_1}$, and $e_{n_2} \to e_{a_2}$, as $\varepsilon \to 0$, into them, we find that eq. (12) goes into eq. (14) and eq. (13) goes into eq. (15). These two remaining expressions yields numerical values for Ω^2 that are in agreement, up to a factor of one-half (because of different definitions) with Jacobi's figures (Lyttleton 1951).

III. EQUILIBRIUM FIGURES

We now analyze our expressions for the angular velocity, and it will prove convenient to unfold the conclusions of such analysis into three major aspects: 1) the existence of equilibrium figures, 2) the existence of a series can be ruled out, and 3) the existence of a lower limit on the quantity ε.

a) *The Existence of Figures*

We have seen that the condition that our body is free from any external pressure [eq. (10)], together with that of the continuity of pressure at each point of the boundary surface between its two homogeneous parts [eq. (11)], leads as a result the two couples of equations [(12), (13)] and [(14), (15)], respectively. Individually, each of these equations expresses the angular velocity of the body in terms of its eccentricities e_{n_i}, e_{a_i}, its relative density ε and the elliptic integrals denoted by F_n, E_n, F_a and E_a. We may summarize them altogether by

$$\Omega^2 = G_i(e_{n_1}, e_{n_2}, e_{a_2}, \varepsilon),$$

with the i subscript running from 1 to 4, i.e., equations (12)–(15). The explicit form of each G_i of such equations may briefly be writen as

$$\varepsilon E_n(x_{1i}) + \varepsilon F_n(x_{2i}) + E_a(x_{3i}) +$$

$$F_a(x_{4i}) + \varepsilon(x_{5i}) + 1(x_{6i}), \qquad (16)$$

in which x_{1i}, x_{2i}, ..., denote certain algebraic expressions in the eccentricities, different from each other and, further, different from one equation to the next. We note, however, that all four terms $F_n(x_{2i})$ [and all four terms $F_a(x_{4i})$] are identical except for an algebraic sign, which changes from one equation to the next.

So far, the existence in principle, of equilibrium figures for our model is clear, since our equations can be seen choosing the relative difference in density ε of the body as a parameter, as a set of four equations involving four variables: $e_{n_1}, e_{n_2}, e_{a_2}$, and Ω^2.

b) *The Exclusion of a Series*

Before going on further, it is well to ask about the consequence that, *a priori*, would mean the insertion of relation (9) (i.e., the dependence among the eccentricities, result of the confocality hypothesis) in our analysis.

In particular, we are interested in knowing if a relationship among the four equations exists so that not all of them are independent. If such a relationship exists, the possibility of a continuum of solutions (a series) is open, since in this case one of the equations could be eliminated and the system, for given ε, has an infinite number of solutions.

Such a continuum of solutions would then represent a whole family of figures with different eccentricites and angular velocities, for each pre-fixed value of the body's densities.

We now proceed to show that no linear dependence exists, excluding therefore the possibility of a series.

The starting point is to assume the existence of a set of factors of a linearity relation among the equations and we show that a contradiction arises.

Calling $q_i (i = 1, 2, 3, 4)$ such factors, the linearity condition (an identity) is expressed by

$$q_1(\Omega^2 - G_1) + q_2(\Omega^2 - G_2) +$$

$$+ q_3(\Omega^2 - G_3) + q_4(\Omega^2 - G_4) \equiv 0$$

Notice that such set of factors has to fulfill the two restritions: i) $q_1 + q_2 + q_3 + q_4 = 0$, and ii) $q_1 - q_2 + q_3 - q_4 = 0$. Restriction i) is a consequence that Ω^2 is an independent variable, and so that

$$(q_1 + q_2 + q_3 + q_4)\Omega^2 = 0$$

in the above identity must be zero. Restriction ii), on the other hand, is justified as follows. Consider expression (16), which represents the typical form of the right-hand side of any of equations (12)-(15). In particular, as was previously quoted, the terms $F_n(x_{2i})$ are, except for an algebraic sign, identical in the four equations, and the same applies to the terms $F_a(x_{4i})$. Therefore we have that, for instance, $F_n(q_1 - q_2 + q_3 - q_4)x_{21} = 0$ and $F_a(q_1 - q_2 + q_3 - q_4)x_{41} = 0$; restrictions *ii*) then follows, since F_n and F_a are independent of each other. Accordingly, the q's are reduced to only two: $q_1 = q_3$ and $q_2 = -q_4$.

From the above reasoning the linearity condition turns to be $q_1(G_1 - G_3) + q_2(G_2 - G_4) = 0$, suggesting that q factors can be found only from solely the differences among equations given by [(12)–(14)] and [(13)-(15)]. Either of these two differences can, with the same notation as that for expression (16), be writen as $\varepsilon E_n(x_1) + E_a(x_3) + \varepsilon(x_5) + 1(x_6)$, because the terms $\varepsilon F_n(x_{2i})$ and the terms $F_a(x_{4i})$ then cancel out. The process of finding the q factors may now be initiated by demanding the terms $1(x_6)$ to vanish, in which case it is found that the q factors must fullfill the ratio

$$\frac{q_1}{q_2} = \frac{(e_{n_2}^2 - e_{n_1}^2)}{(1 - e_{n_1}^2)(e_{n_2}^2 - e_{n_1}^2 e_{a_2}^2)}.$$

These two factors allow the cancellation of the terms $\varepsilon E_n(x_1)$ and the terms $E_a(x_3)$ as well, but they do not cancel the terms $\varepsilon(x_5)$ and, reciprocally, it can be found that the factors which cancel the terms $\varepsilon(x_5)$ do not cancel the rest of them. Hence, we have reached a contradiction, and the assumed existence of a linear dependence among our equations is not obtained.

A series solution for our problem, in the manner of McLaurin's or Jacobi's homogeneous fluids must therefore be ruled out. Stated differently, for the assumed geometry of our model (confocality of the ellipsoids), only one figure (with specific eccentricities) can fit mathematically for each value of

$$\varepsilon\left(= \frac{\rho_n - \rho_a}{\rho_a}\right) .$$

This result, though restricted, is nonetheless surprising for it sharply contrasts with the *total* absence of equilibrium figures for the two-axis model (and in which, as here, $\omega_n = \omega_a \equiv \omega$) (Montalvo, Martínez and Cisneros 1983).

b) *Numerical Results*

In what follows, numerical results are presented for the parameters that describe equilibrium figures. We will see that there is a given range of ε in which no figures exist at all, however.

This is the essential content of Table 1, in which the (unique) set of eccentricities (columns 2, 3, 4, 5) and angular velocity (column 6) that the body attains for each particular value of ε (column 1) are given.

Notice that ε, which measures the relative degree of the body's inhomogeneity, is taken greater tha zero: the values less than -1 must be exclude since $\varepsilon = \rho_n/\rho_a - 1$, and negative densities are n allowed. On the other hand, if $-1 \leq \varepsilon \leq 0$ the it follows that $\rho_a > \rho_n > 0$, but we cast away this cas since in real stars the nucleus is commonly heavie than the atmosphere.

Quite apart from this ad-hoc restriction on we have no hint from previous work either b others or ourselves to expect a limiting value fo ε. Numerical analysis tells us, however, that th existence of figures with $e_{n_1} > e_{a_1}$ and $e_{n_2} > e_a$ [as required by relation (8)] is conditioned to valu of ε greater than 1.1839682 (see next section). Th value corresponds to a situation in which the densi of the interior ellipsoid turns out to be slightly larg than twice the density of the exterior one.

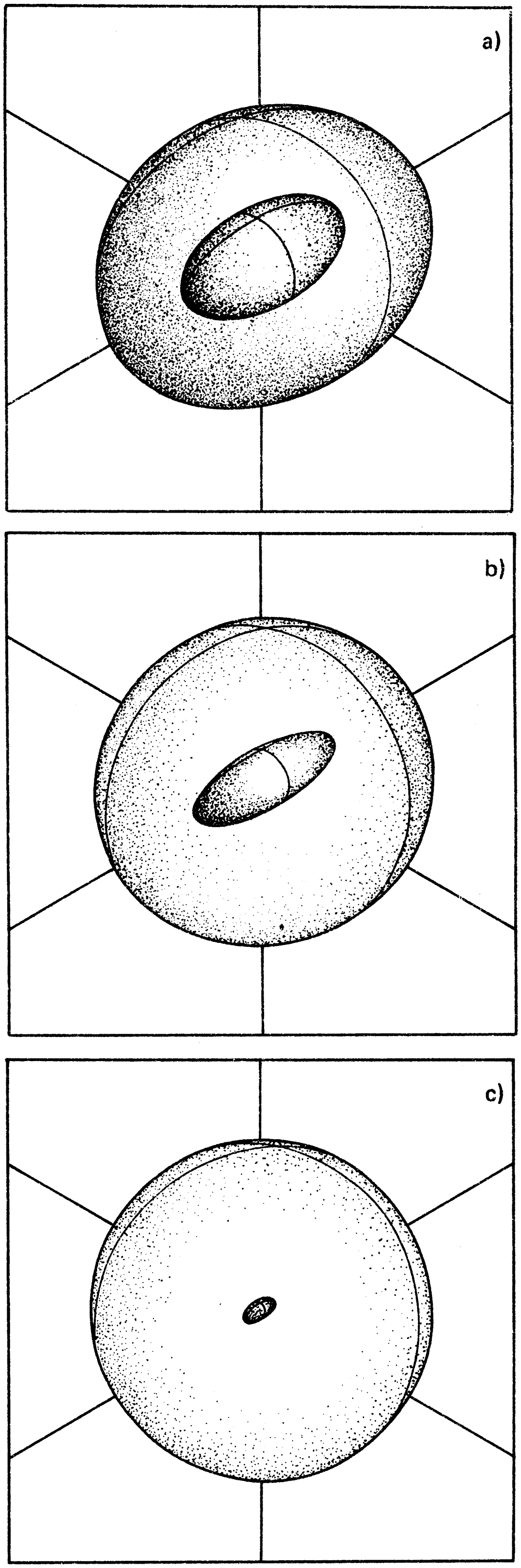
In Figure 1, a scheme of the body's relative shap for each of three different ε values is shown: ($\varepsilon = 2$, which is not very far from the limit, ($\varepsilon = 10$, and (c) $\varepsilon = 50$ (see the correspondin values of eccentricities and angular velocity *italics* in Table 1). Thus, for low ε values (sligl inhomogeneities), but of course greater than th limiting value for which equilibrium figures ca exist, i.e., case (a) the interior ellipsoid possesses rather moderate equatorial and a high meridion degree of flattening. As ε grows to 10, case (b and then to 50, case (c) the equatorial sectic

TABLE 1

PARAMETERS DESCRIBING THE EQUILIBRIUM MODELS OBTAINED FROM EQUATIONS [(12)–(15)]

ε^a	e_{n_1}	e_{n_2}	e_{a_1}	e_{a_2}	$\frac{\omega^2}{4\pi G\rho_a}$	$\frac{T}{GM^2 r^{-1}}$	$\frac{H}{G^{1/2}M^{3/2}r^{1/2}}$	$\frac{C}{2\pi G\rho_a^2 a_{n_1}^2}$	$\frac{C'}{2\pi G\rho_a^2 a_n^2}$
1.1839682	0.6388	0.8986	0.517	0.7272	0.136428	0.0556	0.2423	−1.1011	−1.1318
1.195	0.6029	0.8977	0.4675	0.6961	0.1206	0.0513	0.2269	−1.198	−1.3441
1.2	0.5956	0.8977	0.4576	0.6896	0.1174	0.0503	0.2237	−1.22	−1.3954
1.5	0.468	0.9107	0.2858	0.5562	0.0624	0.0305	0.1638	−1.885	−3.3757
2	*0.393*	*0.929*	*0.1947*	*0.4602*	*0.0375*	0.01973	0.1288	−2.8128	−7.3712
3	0.3197	0.9501	0.1217	0.3618	0.0208	0.0115	0.0975	−4.69	−19.59
4	0.2782	0.9615	0.0889	0.3075	0.0143	0.0081	0.0815	−6.6	−37.69
5	0.25	0.9687	0.0701	0.2719	0.0109	0.0062	0.0714	−8.55	−61.72
10	*0.1792*	*0.9838*	*0.0341*	*0.1876*	*0.0049*	0.0029	0.0483	−18.42	−271.63
15	0.1471	0.9891	0.0225	0.1518	0.0031	0.0018	0.038	−28.37	−631.38
20	0.1278	0.9918	0.0168	0.1309	0.0023	0.0013	0.0334	−38.35	−1141.1
25	0.1145	0.9934	0.0134	0.1167	0.0018	0.0011	0.0297	−48.33	−1800.79
30	0.1047	0.9945	0.0112	0.1064	0.0015	0.00091	0.027	−58.32	−2610.48
35	0.097	0.9952	0.0095	0.0983	0.0013	0.00078	0.025	−68.31	−3570.16
40	0.0908	0.9958	0.0083	0.0919	0.0011	0.00068	0.023	−78.3	−4679.84
45	0.0856	0.9963	0.0074	0.0866	0.001	0.0006	0.022	−88.3	−5939.51
50	*0.0813*	*0.9966*	*0.0066*	*0.0821*	*0.0009*	0.00054	0.02	−98.3	−7349.18
100	0.0576	0.9983	0.0033	0.0579	0.00044	0.00026	0.014	−198.2	−29695.8

a. ε is the relative difference in density between nucleus and atmosphere, e_{n_1}, e_{n_2}, e_{a_1}, e_{a_2} are the equatorial (1) ar meridional (2) eccentricities for nucleus (n) and atmosphere (a). ω, T and H refer to angular velocity, kinetic energy ar angular momentum of the body. C, C' are constants appearing in eqs. (10) and (11) respectively.



g. 1. A schematic view of three figures: (a) for $\varepsilon = 2$,) for $\varepsilon = 10$ and (c) for $\varepsilon = 50$.

slowly tends to a circle whereas the meridional flattening rapidly grows so that the nucleus tends to become a disk. As the exterior ellipsoid, it starts with a low degree of equatorial flattening and a moderate meridional one, which more or less rapidly tends to zero. Thus, the shape of a very inhomogeneous figure resembles a central bulge, much as if the yolks of two fried eggs were placed back to back, surrounded by a nearly spherical much more tenuous mass.

From the above considerations, we may state that although the body's velocity is independent of the size of the body, it does depend on the relative sizes of its two homogeneous parts.

c) *On the Existence of the Limit*

We have established on purely numerical grounds that ε, the quantity that stands for the relative density of our model, is limited to values higher than 1.1839682 (called ε_0 henceforward) for equilibrium figures to exist.

In this section, we look for (under two different approaches) the source of limit ε_0, supporting its existence by the inference that a disrupting mechanism begins to act on the body as soon as ε approaches ε_0 from nearby, higher values. Notice at the top of column 6 that the angular velocity itself has an unusual increase precisely within such interval and we therefore already have a strong hint as to the existence of ε_0, since a very large value of angular velocity means, in effect a disrupting mechanism of the body. Our first approach considers the body's kinetic energy (T) and angular momentum (H). The first quantity, when normalized to GM^2r^{-1} (column 7) is given by

$$\frac{T}{GM^2r^{-1}} = \left(\frac{3}{10}\Omega^2\right)$$

$$\times\left[1+\varepsilon\left(\frac{e_{a_1}}{e_{n_1}}\right)^5 \frac{(2-e_{n_1}^2)(1-e_{n_1}^2)^{1/2}(1-e_{n_2}^2)^{1/2}}{(2-e_{a_1}^2)(1-e_{a_1}^2)^{1/2}(1-e_{a_2}^2)^{1/2}}\right]$$

$$\times\left[1+\varepsilon\left(\frac{e_{a_1}}{e_{n_1}}\right)^3 \frac{(1-e_{n_1}^2)^{1/2}(1-e_{n_2}^2)^{1/2}}{(1-e_{a_1}^2)^{1/2}(1-e_{a_1}^2)^{1/2}}\right]^{-2}$$

$$\times\left[\frac{(2-e_{a_1}^2)}{(1-e_{n_1}^2)^{1/6}(1-e_{n_2}^2)^{1/6}}\right],$$

while H^2 when normalized to GM^3r (column 8), is given by

$$\frac{H^2}{GM^2 r} = \left(\frac{3}{25}\Omega^2\right) \frac{\left(2 - e_{a_2}\right)^2}{\left(1 - e_{a_1}^2\right)^{2/3}\left(1 - e_{a_2}^2\right)^{2/3}}$$

$$\times \left[1 + \varepsilon \left(\frac{e_{a_1}}{e_{n_1}}\right)^5 \frac{\left(2 - e_{n_1}^2\right)\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{\left(2 - e_{a_1}^2\right)\left(1 - e_{a_1}^2\right)^{1/2}\left(1 - e_{a_2}^2\right)^{1/2}}\right]^2$$

$$\times \left[1 + \varepsilon \left(\frac{e_{a_1}}{e_{n_1}}\right)^3 \frac{\left(1 - e_{n_1}^2\right)^{1/2}\left(1 - e_{n_2}^2\right)^{1/2}}{\left(1 - e_{a_1}^2\right)^{1/2}\left(1 - e_{a_2}^2\right)^{1/2}}\right]^{-3},$$

where M is the body's mass and $r = \left(a_{a_1} a_{a_2} a_{a_3}\right)^{1/3}$.

For completeness of this first approach concerning the existence of ε_0, we calculate the constants which appear in eq. (10), C, and in eq. (11), C′. If the existence of ε_0 is taken as granted, then C and C′ can be expected to show a tendency to vanish. Such tendency would mean on C, the approach to the final body's bound states; on C′, on the other hand, it would mean the violation of the continuity of pressure at the interface nucleus-atmosphere.

The values of C and C′, when normalized to $2\pi \cdot G\rho_a^2\, a_{n_1}^2$ are given in columns 9 and 10.

The manifestation of the previously quoted abnormal behavior within a small interval of ε values greater than, but in the neighborhood of ε_0, is clear in all of the four analyzed quantities: T, H, C and C′ and the two constants have, in addition, a remarkable tendency to vanish, in agreement with our expectations.

We now try to understand the existence of limit ε_0 from a different point of view. We believe that, as Ω^2 grows (and ε decreases, see Table 1, columns 1 and 6), a situation will finally be reached where it is impossible to fulfill at least one of the equations (12)–(15), because Ω^2 overrides the remaining part of the equation [see any of eqs. (12)–(15)] and therefore no model is possible. In order to see in which equation the unbalance occurs, we proceeded as follows. We took Ω^2 slightly greater than the limiting value of Ω_0^2 (that corresponds to ε_0). We then (artificially) took Ω^2 smaller in the first equation of the system, by subtracting from it a small quantity, and found that the altered system admitted no solution. I[n] the same way, we proceeded with the second an[d] third equations without success. However, whe[n] the fourth equation was changed, a solution of th[e] system could be found, showing us that in thi[s] equation arose the unbalance, responsible for th[e] limit. We recall that this equation i.e., eq. (15) corresponds to the equality of the second and thir[d] ratios of eq. (6), as applied to the quantity φ', buil[t] from the condition of continuity of pressure [eq. (11)] at the nucleus-atmosphere border. Going on [a] little further, let us write the parent equation of e[q.] (15) [see eq.(6)] as

$$\frac{\left(F_g + F_c\right)x_2}{Ax_2} = \frac{\left(F_g\right)x_3}{Bx_3},$$

where F_g stands for gravitational force, F_c for cen[-]trifugal force, A and B being constants (notice tha[t] there is no third component of centrifugal force pre[-]sent since our body rotates around the third axis). Therefore, the equality sign of the above equatio[n] which applies only when the body is in relative hy[-]drostatic equilibrium, no longer applies in conse[-]quence of the relative large increase in angular velo[-]city, ε approaches ε_0, to which the centrifugal forc[e] is directly proportional. Equivalently, the existenc[e] of ε_0 may be attributable to the violation of the equi[-]librium condition given by eq. (11). Therefor[e] there is a total absence of equilibrium models fo[r] ε values smaller than ε_0.

This work was supported in part by FAI-UASLP.

REFERENCES

Chandrasekhar, S. 1969, *Ellipsoidal Figures of Equilibriu[m]* (Yale University Press).

Landau, L.D. and Lifshitz, E.M. 1959, *Fluid Mechanic[s]* (Pergamon Press).

Lyttleton, R.A. 1951, *The Stability of Rotating Liquid Masse[s]* (Cambridge).

MacMillan, W.D. 1958, *Theoretical Mechanics: The Theory [of] the Potential*, (New York: Dover Publications, Inc.

Milne-Thomson, L.M. 1968, *Theoretical Hydrodynamic[s]* (The MacMillan Press).

Montalvo, D., Martínez, F.J., and Cisneros, J. 1983, *Re[v.] Mexicana Astron. Astrofis.*, **5**, 293.

J. Cisneros, F.J. Martínez, and D. Montalvo: Universidad Autónoma de San Luis Potosí, Instituto de Física, Alvar[o] Obregón No. 64, 78000 San Luis Potosí, S.L.P., México.