:ev. Mexicana Astron. Astrof., **21**, 569-580 (1990)

REGULAR MOTIONS OF RESONANT ASTEROIDS

(Invited Talk)

S. Ferraz-Mello

Instituto Astronômico e Geofísico, USP, Brazil

RESUMEN. Se revisan resultados analíticos relativos a soluciones regulares del problema asteroidal elíptico promediados en la vecindad de una resonancia con Júpiter. Mencionamos la ley de estructura para libradores de alta excentricidad, la estabilidad de los centros de liberación, las perturbaciones forzadas por la excentricidad de Júpiter y las órbitas de corotación.

ABSTRACT. This paper reviews analytical results concerning the regular solutions of the elliptic asteroidal problem averaged in the neighbourhood of a resonance with Jupiter. We mention the law of structure for high-eccentricity librators, the stability of the libration centers, the perturbations forced by the eccentricity of Jupiter and the corotation orbits.

Key words: **ASTEROIDS**

. INTRODUCTION. MEAN-MOTION AND SECULAR RESONANCES

The study of collective phenomena in the Dynamics of the Solar System gained momentum in the past ecade with the increased possibility of utilization of fast computers and new techniques of propagation of orbits, as ·ell as, in the particular case of planetary rings, with the availability of precise space and ground-based observations. mportant advances were recorded concerning the dynamics of the asteroidal belt, planetary rings, meteor swarms, the)ort cloud and the planetesimals embedded in the pre-planetary nebula.

The interest in the asteroidal belt is due not only to its cosmogonical importance but also to the fact that ndividual asteroids may be seen as probes moving in the phase space of a well-defined dynamical system, providing vidence for new dynamical phenomena. Among the asteroids, the role played by resonances with Jupiter was unravelled y the first studies on the distribution of their orbits, when less than one hundred asteroids were known. The asteroids f the main belt avoid moving in resonance with Jupiter, leading to well-marked gaps in their distribution at periods /2, 2/5, 3/7 and 1/3 of the period of Jupiter. On the other hand, almost all asteroids found moving outside the main elt move *in* resonance with Jupiter. They are the *Hildas* and (279) Thule, whose periods are respectively 2/3 and 3/4 f the period of Jupiter, and the Trojans, whose orbits are oscillations about Jupiter's Lagrangian points L_4 and L_5, hus having, in the average, the same orbital period as Jupiter. All these features are easily visible in Figure 1.

The gaps in the main belt are called ***Kirkwood gaps*** and are usually identified by the rate of the corresponding orbital mean motion to that of Jupiter 2/1, 5/2, 7/3 and 3/1 (instead of the ratio of the periods). In the same ·ay, the resonances where lie the *Hildas* and (279) Thule are known as 3/2 and 4/3. The *Hildas*, (279) Thule and the 'rojans are not the only asteroids known as moving in resonance with Jupiter. In fact, there are also a few asteroids noving in resonance with Jupiter inside the Kirkwood gaps. They are the *Alindas*, in the gap 3/1, and the *Griquas*, in he gap 2/1. These asteroids present the kinematical behaviour known as libration. A librating asteroid in the resonance $p+q)/p$ is an asteroid whose motion is such that the angle

$$\theta = (p+q)\lambda_{jup} - p\lambda - q\varpi \qquad (1)$$

scillates about a constant value (λ and λ_{jup} are the mean longitudes of the asteroid and Jupiter, respectively, and ϖ is he longitude of the perihelion of the asteroid's orbit).

We may also mention some asteroids moving just outside the gap 2/1 – which we could call *Sibyllas*, ollowing the widely used rule of giving to a group the name of its first member known – for which the angle θ librates bout π. While these asteroids present the same kinematical characteristic of the resonant asteroids mentioned before, heir location in the phase space is such that many authors do not include them among the resonant asteroids (see for instance Poincaré, 1902). In fact their orbits may be obtained as analytic continuation of orbits in which the angle θ circulates, thus, without crossing any true bifurcation. These asteroids are also known as apocentric or small-eccentricity librators, as their eccentricities are always very small (less than 0.1), in contrast with the librators mentioned before whose eccentricities reach values as high as 0.6.

Other asteroids are mentioned, sometimes, as linked to one of the other resonances associated with Kirkwood gaps, but the librating character of their orbits were not confirmed in a definitive way.

Just for sake of completeness we also mention that, in the past, the names of the asteroids (108) Hecuba and (46) Hestia were associated to the resonances 2/1 and 3/1 respectively. However, the actual orbit of these two asteroids lie on the left of the corresponding gaps and are such that the angle θ circulates.

The resonances just considered are known as mean-motion resonances. They are not the only kind of resonance to affect the motion of the asteroids. There are also the secular resonances which occur when the rate of the motion of the longitude of the perihelion, or that of the ascending node, of the asteroid's orbit equals one of the eigenfrequencies of the linear theory of the secular motion of the planets. Given the location of the asteroidal belt three 1/1 secular resonances are important, two of them involving the asteroid perihelion and the eigenfrequencies ν_5 and ν_6 associated with the motion of the perihelion of Jupiter and Saturn, respectively. The other involves the node of the asteroid's orbit and the eigenfrequency ν_{16} associated with the joint motion of the nodes of the orbits of Jupiter and Saturn over the invariable plane of the Solar System. The secular resonances also provoke gaps in the distribution of the asteroids. However, as the motion of the perihelion and node of the asteroids is strongly dependent on the eccentricity and inclination of the orbit, these gaps are apparent only in tridimensional distributions (Williams and Faulkner, 1981). Figure 2 shows the distribution of the mean motions and inclinations of the asteroids with eccentricities less than 0.3. The locations of the secular resonances ν_5,ν_6 and ν_{16} are indicated by dashed lines, and the role played by them in the distribution is evident.

The numerical integration of the equations of the motion of fictitious resonant asteroids, in all these resonances, has shown the existence of motions called ergodic, stochastic or, more recently, chaotic (see for example Giffen, 1973; Froeschlé and Scholl, 1976, 1981 and Wisdom, 1987). However, the numerical integration of orbits of actual asteroids in resonant motions shows that most of them are moving in regions where the solutions of the elliptic asteroidal problem, averaged over the periods of the order of the synodic period of the asteroid and Jupiter or, in the case of the *Trojans*, of the order of the sidereal period of these bodies, are regular. at least for times of the order of the integration intervals (less than 10^6 years).

The regular motions of the asteroids in mean-motion resonance with Jupiter, the *Trojans* excepted, are reviewed in the next sections.

II. HIGH-ECCENTRICITY LIBRATIONS

The regular motions of the asteroids in the resonances 3/1, 2/1, 3/2 and 4/3 are oscillations about stable

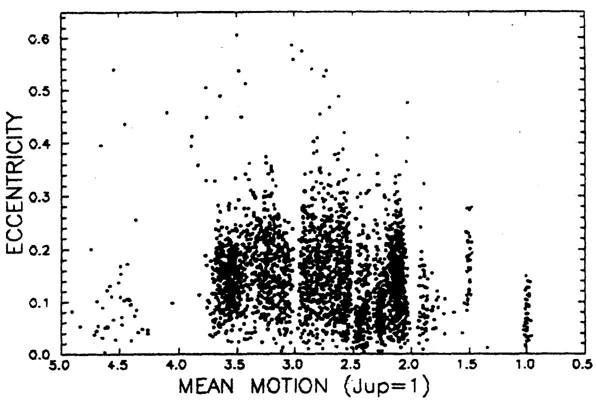


Figure 1. Distribution of the mean motions and eccentricities of the asteroids, showing the gaps at the mean motions 3/1, 5/2, 7/3, 2/1 and the groups at the mean motions 3/2 and 1.

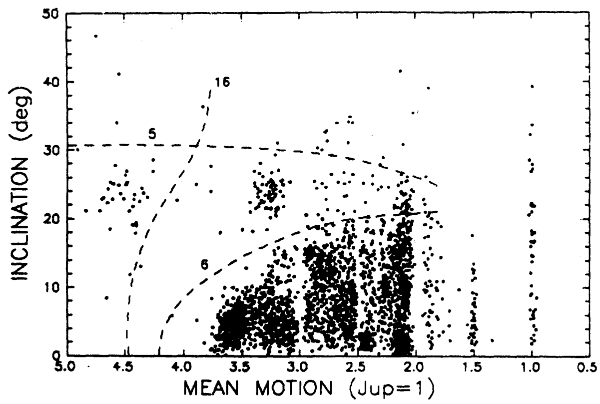


Figure 2. Distribution of the mean motions and inclinations of the asteroids with $e < 0.3$ and the positions corresponding to the secular resonances ν_5, ν_6 and ν_{16} for $e = 0.1$.

periodic solutions of the averaged asteroidal problem known as *libration centers* (see Fig. 10). The libration centers have, generally, high-eccentricity (up to 0.6 in some cases) and the construction of analytical models for studying the oscillations around them, the so-called *librations*, is impaired by the poor convergence of the classical series of Celestial Mechanics. The only successful existing models for the study of the properties of these motions are due to Ferraz-Mello (1988) and Morbidelli and Giorgilli (1989). In both cases the classical expansions were discarded and canonical models were constructed using the expansion of the potential of the disturbing forces in Taylor series about the libration centers, averaged over the synodic longitude. The expansion used by Morbidelli and Giorgilli is fully numerical while that used by Ferraz-Mello is semi-analytical, the coefficients being obtained by numerical quadrature of functions whose analytical expressions are given. If the adopted angle variables are

$$\sigma = \frac{p+q}{q}\lambda_{jup} - \frac{p}{q}\lambda - \varpi$$
$$\sigma_1 = \frac{p+q}{q}\lambda_{jup} - \frac{p}{q}\lambda - \varpi_{jup} \tag{2}$$
$$Q = \frac{\lambda - \lambda_{jup}}{q}.$$

Ferraz-Mello planar expansions for the potential of the forces disturbing the motion of an asteroid in the resonance $(p+q)/p$ have the form

$$\begin{aligned} R = R_0 + R_1(k-k_c) + R_2(h-h_c) + R_3(k-k_c)^2 + R_4(k-k_c)(h-h_c) + R_5(h-h_c)^2 \\ + \{R_6 + R_7(k-k_c) + R_8(h-h_c)\}\, e_{jup}\cos\sigma_1 + \{R_9 + R_{10}(k-\kappa_c) + R_{11}(h-h_c)\}\, e_{jup}\sin\sigma_1 \\ + \{R_{12} + R_{13}\cos 2\sigma_1 + R_{14}\sin 2\sigma_1\}\, e^2_{jup} + \ldots.. \end{aligned} \tag{3}$$

(see Ferraz-Mello, 1987b, 1989a; Ferraz-Mello and Sato, 1990) where the variables k and h are defined by

$$k + \imath h = e.\exp \imath\sigma. \tag{4}$$

e and e_{jup} are the orbital eccentricities of the asteroid and Jupiter. The coefficients R_j are functions of the semi-major axes a, a_{jup} and of the parameters k_c, h_c, values of k, h at the chosen libration center. The averaging of the coefficients R_j over Q is obtained by using elementary procedures of numerical quadrature (not to be confused with the numerical integration of the differential equations). It is worthwhile noting that all calculations for obtaining these coefficients and their averages are done without using any expansion in power of the eccentricities and are valid for all eccentricities.

We note that the averaging used by Morbidelli and Giorgilli is a true averaging of the differential equations, founded on a Lie-series perturbations theory (see Ferraz-Mello, 1989b) performed up to the square of the disturbing masses, while Ferraz-Mello averaged the right-hand side of the equations only (scissors' averaging), which corresponds to the exact averaging at an order less than the square of the disturbing mass of Jupiter (we recall that this order depends on the particular problem studied since, generally, fractional orders are introduced when studying resonances).

When $e_{jup} = 0$ the averaged equations of the motion have periodic solutions. The locus of these solutions in the plane $(\bar{a}, \bar{e})$ (the overlines indicate averaging over Q) was called *law of structure* in Ferraz-Mello (1988) and its analytical expression, valid for any eccentricity, was given. This law was, later on, compared to the results arising from the numerical integration of the equations of the planar motion of an asteroid (non-averaged) showing full agreement (Vienne, 1987, fig.12).

The law of structure is shown in Figures 3-6 for the resonances 2/1, 3/2 and 3/1. We note that, in the first-order resonances 2/1 and 3/2, the law of structure has two branches. The main branch corresponds to solutions for which $\sigma = 0$ and is called *pericentric* since it corresponds to solutions for which the asteroid is at the pericenter of its orbit at the moments of conjunction with Jupiter. The other branch, said *apocentric* for a similar reason, corresponds to solutions for which $\sigma = \pi$. Figures 3-6 show only the sections of these branches corresponding to orbits stable with respect to planar displacements. The recent work of Morbidelli and Giorgilli consider the full spatial motion of the asteroids and the stability of these solutions was studied also with respect to displacements perpendicular to the plane of the motion. They introduced a parameter α, roughly equal to $H_z/max(H_z - \overline{H_z})$ where $H_z(R,\omega)$ is the main contribution of the space action-angle variables R,ω to the averaged Hamiltonian and the overline indicates averaging over the angle ω. If $\alpha > 1$ the curves $H_z = const.$, in the plane of the polar coordinates R,ω, are closed and the origin represents a stable solution. However, in order to warrant stability over long periods of time this ratio must be large enough since, otherwise, the non-considered terms cannot be considered as a small relative perturbations, and it is likely that a rigorous

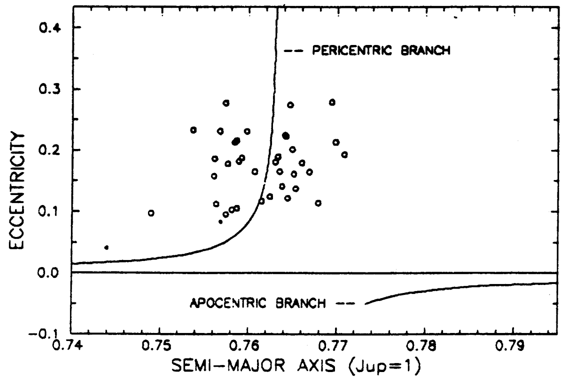


Figure 3. Law of Structure for the resonance 3/2. The circles represent the elements of the Hildas in 1987. The dots are the circulators 334 (Chicago) and 1256 (Normannia).

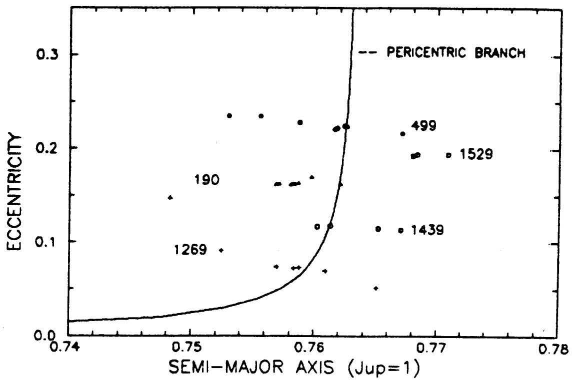


Figure 4. Observed libration of some Hildas. The points represent the orbital elements as actually determined from the observations. All, except (1269) Rollandia, evolve with increasing semi-major axes.

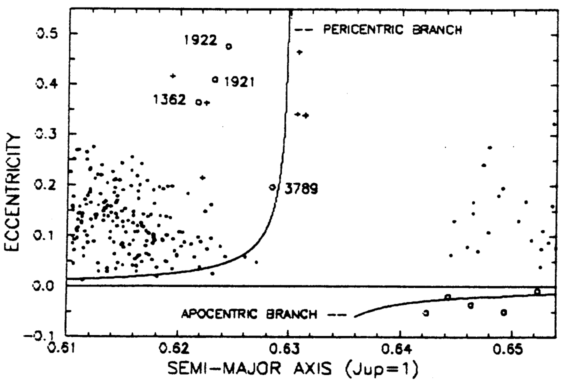


Figure 5. Law of Structure for the resonance 2/1. The dots represent the osculating elements of asteroids close to the gap. The librators are shown by circles. The crosses indicate the elements of the pericentric librators as determined from the observations, in several occasions, since their discovery.

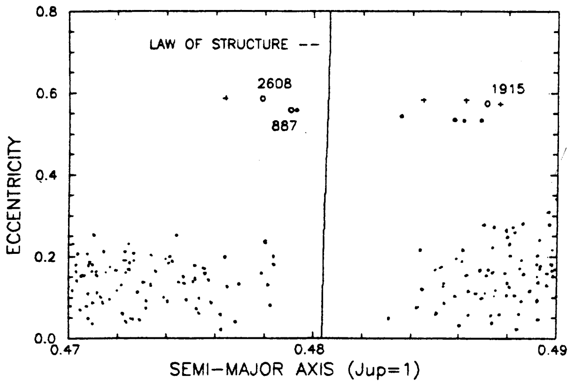
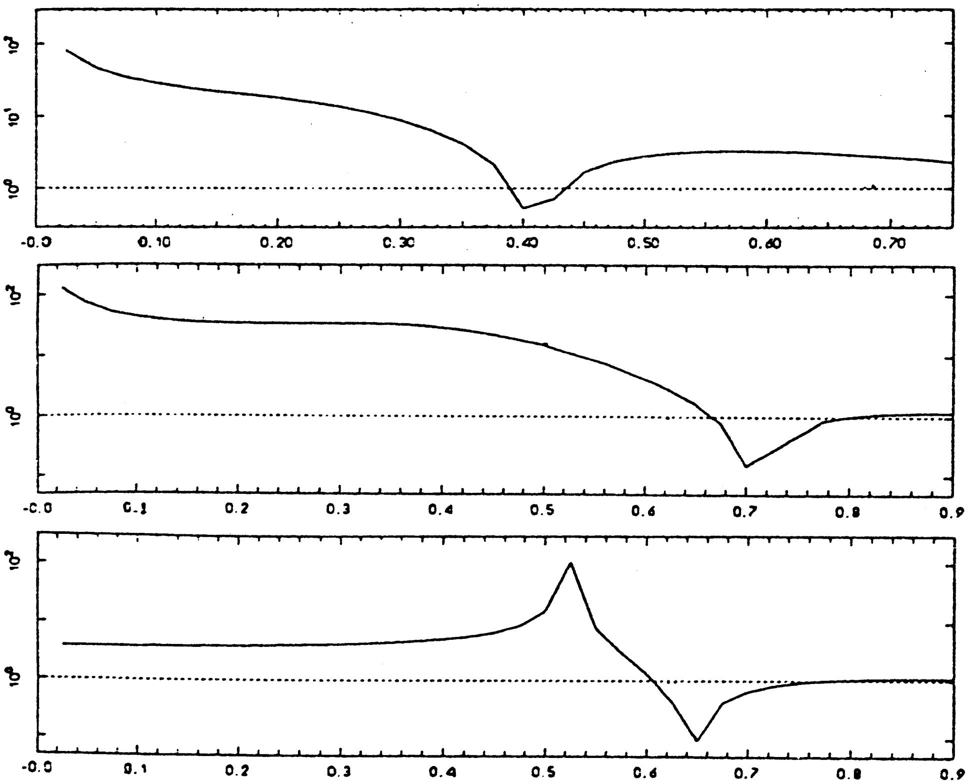


Figure 6. Law of Structure for the resonance 3/1. The dots represent the osculating elements of asteroids close to the gap. The librators are shown by circles. The crosses and large dots indicate the elements of the librators as determined from the observations, in several occasions, since their discovery.

analysis would reveal that diffusion may happen in short times. Thus, if α is not large enough, say $\alpha < 10$, Morbidelli and Giorgilli expect the solutions to be practically unstable over the long scale of the problem under consideration. The values of α computed by them, for the resonances 2/1, 3/2 (pericentric branches) and 3/1, are shown in Figure 7.

Figures 3-6 also show the actual osculating elements of some asteroids. In Figure 3 the dots represent the osculating elements a and e of all asteroids, in the range $0.74 < a/a_{jup} < 0.795$, numbered before November, 1985, according to the *Efemeridy Malikh Planet* for 1987. They are almost equally distributed on both sides of the pericentric branch. Their actual values oscillate around one center on the pericentric branch. In the 250–270 years of a libration, the eccentricity and the semi-major axis of a librating asteroid undergo a complete oscillation about the values corresponding to a libration center. This scenario does not exclude the presence of asteroids in the structural gap 0.763–0.773 a_{jup} (3.970–4.019 A.U.) where no stable solution exists. It just prevents them from remaining stationary there. This structural gap may be crossed and, indeed, at any time, some 15 to 20 asteroids of the quoted list may be found in the gap, those of greater libration amplitude being able to cross the whole gap before returning; this is the case of the asteroids (1269) Rollandia, (1529) Oterma and (1748) Mauderli, which, according to Nakai and Kinoshita (1986), may reach 4.05, 4.03 and 4.03 A.U., respectively. We note that no asteroid is known in the region corresponding to the apocentric branch of the resonance 3/2.



ECCENTRICITY OF LIBRATION CENTER

Figure 7. Stability ratio α. The motion is unstable for displacements orthogonal to the orbital plane for $\alpha < 1$. Stability over long spans of time is only granted if $\alpha >> 1$. (Morbidelli and Giorgilli, 1989). From top:(a) Resonance 3/2; (b) Resonance 2/1); (c) Resonance 3/1.

In Figure 4, we show the actual variation of the orbital elements of 5 *Hildas* (out of 45 included in the 1989 ephemerides): (190) Ismene, (499) Venusia, (1269) Rollandia, (1439) Vogtia and (1529) Oterma, according to the data published yearly, from 1881 to 1918, in the *Berliner Astronomisches Jahrbuch*, from 1912 to 1945, in the *Bahnelemente und Oppositionsephemeriden der Kleinen Planeten* and, from 1945 to 1989, in the *Efemeridy Malikh Planet.* All points represent orbits actually determined from the observations (orbits obtained just by numerical or analytical propagation were discarded). This figure provides an observational confirmation of the results of the theory of high-eccentricity librators, showing that the orbital semi-major axis of these librators may have large variations (see Ferraz-Mello, 1988, figure 8). These variations, up to 3 percent, show the impossibility of considering a constant semi-major axis during a whole libration, as an approximation, as done in past theories.

In Figure 5, the osculating elements of the asteroids in the range $0.61 < a/a_{jup} < 0.654$ are represented. Most of them, shown by dots, are not oscillating about a libration center and their perihelions circulate with respect to the lines of conjunction with Jupiter (that is, the angle σ circulates); their semi-major axes are almost constant. The circles on the left side are the *Griquas*: (1362) Griqua, (1921) Pala, (1922) Zulu and (3789) Zhongguo (ex-1928 UF). The circles on the right side are the apocentric librators, from left to right, (529) Rezia, (401) Ottilia, (1177) Gonnessia, (1266) Tone and (168) Sibylla. The apocentric branch is drawn on the negative half-plane since it corresponds to $\sigma = \pi$ (the vertical axis representing, in fact, $e.\cos\sigma$, instead of the eccentricity). For the dots, the locations on the positive or negative half-plane is meaningless since their corresponding σ circulate. As for the *Hildas*, the *Griquas* have large variations in their semi-major axes and the crosses indicate the elements of (1362) Griqua, (1921) Pala, (1922) Zulu and (3789) Zhongguo, as determined from the observations since their discoveries. The identification of these data in the figure may be easily done by recalling that the eccentricity of high-eccentricity librators have only slight variations in one libration period.

Figure 6 show the actual osculating elements of the asteroids whose semi-major axis is in the range $0.47 < a/a_{jup} < 0.49$. Most of them, represented by dots, have relatively small eccentricity and are not oscillating about a libration center (the corresponding angles σ circulate). The circles indicate the *Alindas*: (887) Alinda, (1915) Quetzálcoatl and (2608) Seneca. In this case, the law of structure has only one stable branch, corresponding to $\sigma = \pi/2(mod\ \pi)$, shown in the figure. Crosses and large dots indicate the elements of these asteroids as determined from the observations. It is worthwhile to mention the observed evolution of the orbital elements of (887) Alinda from one side of the resonance to the other, since its discovery in 1918 (large dots).

III. THE ROLE OF THE ECCENTRICITY OF JUPITER

When the orbit of Jupiter is considered circular, the main features of a librating asteroidal orbit are a free oscillation about a periodic orbit (the libration center) with a period of a few centuries (roughly 250 to 450 years) and short period fluctuations.

The consideration of the non-zero eccentricity of Jupiter introduces, in the motion of high-eccentricity librators, an additional long-period oscillation associated with the motion of the line of conjunctions about the perihelion of Jupiter's orbit. The period of this oscillation is generally in the range 1,000–10,000 years (see, for instance, Nakai and Kinoshita, 1986). In the apocentric, or small-eccentricity librators, this period is smaller and close to the libration period, giving rise to very-long-period beats.

This modelling is a direct consequence of the use of Sessin's transformation, a transformation allowing the abridged equations of the averaged elliptic asteroidal problem to be reduced to that of the circular one (see Ferraz-Mello, 1987a). Geometrically, this composition of motions is shown in Figure 8, in two different reference frames. In the reference frame giving $z = e.\exp\imath\sigma$ (fig. 8a), we have

$$z = k_c + A\exp\imath\alpha + Be_{jup}\exp\imath\sigma_1 \tag{5}$$

where k_c is the eccentricity of the libration center (assumed in this figure as a pericentric one, i.e., such that $\sigma = 0$), A and α are the amplitude and phase of the libration (A is not a constant but an even 2π-periodic function of α) and Be_{jup} the amplitude of the term forced by the ellipticity of Jupiter's orbit.

To change to the more usual reference frame giving $\zeta = e.\exp\imath(\varpi - \varpi_{jup})$, we note that $\zeta = z^*\exp\imath\sigma_1$ (z^* is the complex conjugate of z) and obtain

$$\zeta = Be_{jup} + k_c\exp\imath\sigma_1 + A\exp\imath(\sigma_1 - \alpha) \tag{6}$$

(see fig. 8b).

An example of this composition, in the reference frame of z, is given in figure 9 which shows a planar averaged modelling of the libration of (279) Thule under the action of Jupiter (Tsuchida, 1990). It is composed of a banana-shaped free oscillation about the center $k_c \approx 0.07$, with period 190 years, and a circular component forced by the ellipticity of Jupiter with an amplitude of the order of k_c and a period of 450 years. Figure 10 shows the complete solution during almost the same time as in Figure 9. The differences between the modelled and actual motions are mainly due to the averaging, since the closeness of the orbits of (279) Thule and Jupiter gives rise to very large short-period oscillations (see Tsuchida, 1990, fig.6).

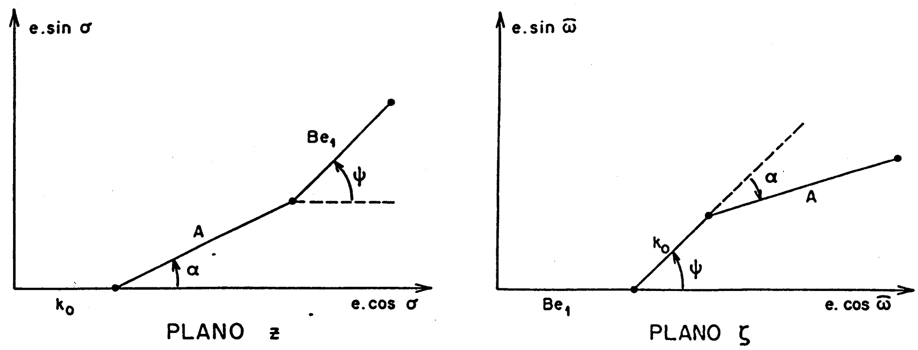


Figure 8. Composition of the main oscillations due to resonance in two different reference frames: (a) $z = e.\exp\imath\sigma$; (b) $\zeta = e.\exp\imath(\varpi - \varpi_{jup})$; N.B. $k_0 \equiv k_c, e_1 \equiv e_{jup}$.

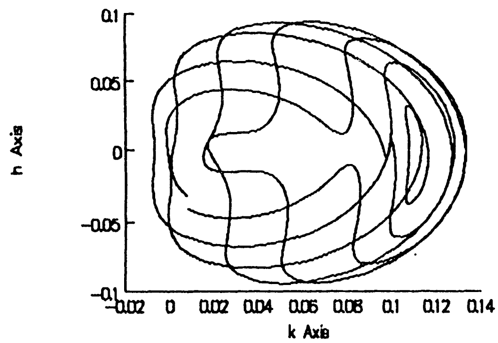


Figure 9. Averaged libration of 279 (Thule) as modelled with the theory of high-eccentricity librations (Tsuchida, 1990) during 2200 years.

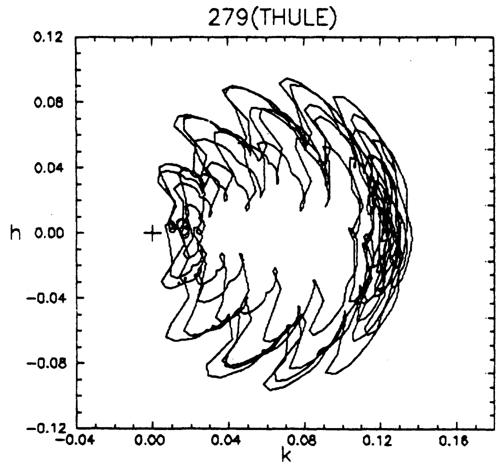


Figure 10. Complete libration of 279 (Thule) during 2134 years. The cross shows the origin of the plane (k,h).

Figure 11 shows the value of the forced amplitude Be_{jup} given in the theory of high-eccentricity librations ɜrraz- Mello, 1988) and, for sake of comparison, those obtained by Schubart (1968) through a numerical averaging and egration of the equations of the motion. We point out that the law giving this second forced mode, due to the ellipticity Jupiter's orbit, in the theory of high-eccentricity librations, is diferent from the similar one derived by Greenberg and ınklin (1975) in the study of 2:1 apocentric librations. However, a higher-order theory allows the conciliation of both ults, our result being valid for high-eccentricity librations ($k_c > 0.1$), in the limiting case where $\dot{\sigma}_1 \approx 0$, while the ult given by Greenberg and Franklin is valid for small-eccentricity librations, where $\dot{\sigma}_1$ is finite; we may note that ≈ 0 when the averaged semi-major axis is close to the critical value corresponding to the vertical asymptote to the ricentric branch. For libration centers with small eccentricity, apocentric or pericentric, a is far enough of these critical ues and $\dot{\sigma}_1$ is small but finite.

In fact, this second law is weaker than the law of structure, since it is obtained from a reduction of the miltonian strongly influenced by the hypotheses made on the order of magnitude of the quantities involved and is id only when $\dot{\sigma}_1$ is close to zero while the law of structure was obtained from a Hamiltonian where only the terms tored by e^2_{jup} were neglected.

Figures 12 give this law in a wider region for the resonances 3/2 and 3/1. In both cases the abscissas are ɜ eccentricities of the libration centers and the two continuous lines give the minimum and maximum absolute values

$$z_0 = k_c + Be_{jup} \exp \imath\sigma_1 \tag{7}$$

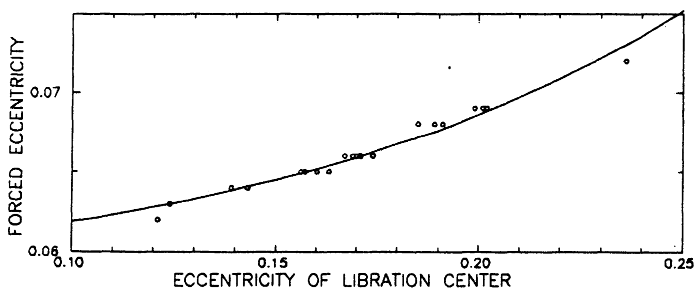


Figure 11. Amplitude of the forced mode of oscillation induced by the eccentricity of Jupiter. Solid-line: high-eccentricity libration model. Dots: Schubart's results from numerical averaging and integration.

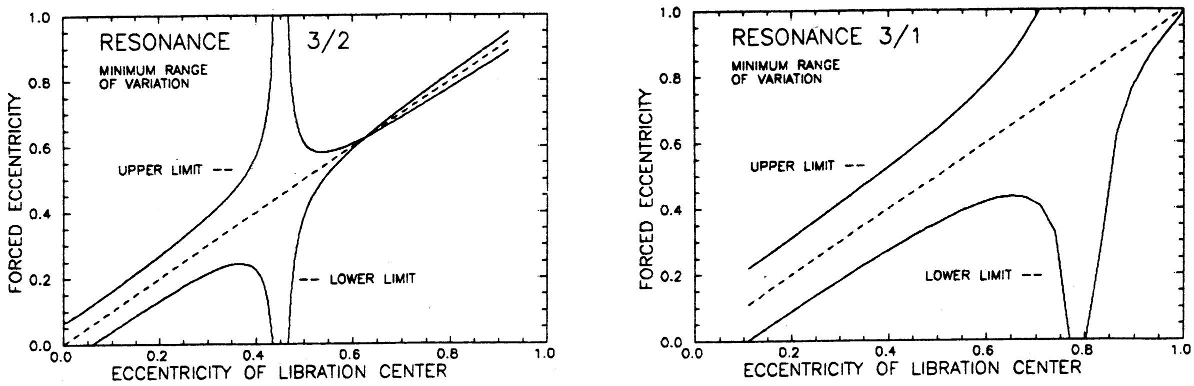


Figure 12. Minimum range of variation of the eccentricity of a resonant asteroid forced by the eccentricity of Jupiter. (a) Resonance 3/2; (b) Resonance 3/1.

as σ_1 circulates. The fact that in both cases the upper limit reaches the level 1.0 means that the corresponding orbits may reach $e = 1$ and escape from its original region in the Solar System, even in the case of a libration with a very small amplitude ($A \approx 0$).

In fact, the theory of high-eccentricity librations is no longer valid when $| z - k_c |$ becomes large. Anyway, this result shows the existence of a barrier which must play a role in the actual distribution of the asteroids. It is meaningful that, in the resonance 3/2, all asteroids always have eccentricities below 0.34 (see Nakai and Kinoshita, 1986), far below the region where a sharp increase of e is possible. In the resonance 2/1 this barrier is higher. The sharp increase occurs only after $k_c = 0.6$, a value also larger than the observed eccentricities of the *Griquas*. Also in the resonance 3/1 the observed values are less than 0.6, i.e., on the left of the sharp increase of the upper branch of Fig. 12(b).

The double resonance represented by $\dot{\sigma} = \dot{\sigma}_1 = 0$ is the mechanism responsible for the limitation of the mean orbital eccentricities of librating asteroids. We note that it may not act alone, but in conjunction with the instability of librations to displacements perpendicular to the orbital plane discussed in the previous section, because of the almost coincidence between the forbidden intervals provided by the eccentricity forced by Jupiter with those where $\alpha < 0$ in Fig. 7.

A final remark must be done concerning the resonances with $q > 1$. In these higher-order resonances the transformation of Sessin is valid for $e >> e_{jup}$ only. Otherwise more than three components should be considered in Fig. 8.

IV. PLANAR COROTATIONS

An asteroid is said to undergo a corotation when it is simultaneously involved in both a mean-motion and a secular resonance. The bases of the theory of the planar corotations are the same of the theory of high-eccentricity librations, with the only difference that now the expansions are done around the exact corotation (or *corotation center*) instead of the libration centers. The results obtained so far show the existence of linearly stable and linearly unstable corotation centers at finite eccentricities.

Corotation centers are orbits given by the equilibrium solutions of the averaged equations of motion. Ferraz-Mello (1990) has shown that a necessary condition for an orbit to be a corotation center is

$$\begin{aligned} \sigma &= 0 \qquad && (mod\ \pi/q) \\ \sigma_1 &= \sigma && (mod\ \pi). \end{aligned} \tag{8}$$

The conditions for having a corotation center are far more strict than those given by the law of structure for the libration centers. There, we have $\sigma = 0(mod\ \pi/q)$ only, leading to a law relating the semi-major axis and the eccentricity. Here, we have a second relation to be fulfilled, corresponding to a second, independent, resonance, which leads to a new law relating the eccentricities of the orbits of the asteroid and Jupiter.

Figure 13 shows the exact corotation solutions for q=1 and p=1,2,3,4,5. The vertical axis represents

$k = e.\cos\sigma$ and the horizontal axis represents $s_1 = e_{jup}\cos\sigma_1$. Thus, in both cases we have a kind of algebraic value of the eccentricity whose absolute value is the orbital eccentricity and whose signal is determined by the position of the corresponding body – the asteroid or Jupiter – at the time of their symmetric conjunction. It is positive if the body is at the pericenter of its orbit and negative if it is at the apocenter. The sections indicated with dots are those where the solutions are linearly stable.

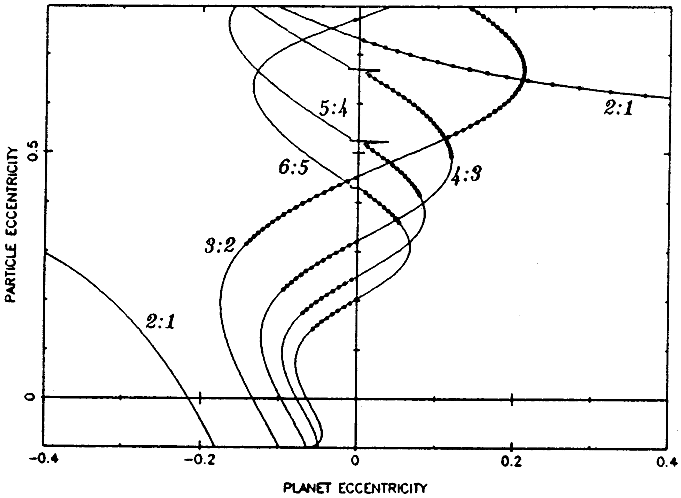


Figure 13. Exact corotation solutions for the resonances 2/1, 3/2, 4/3, 5/4 and 6/5. The vertical axis is $e\cos\sigma$ and the horizontal axis is $e_{jup}\cos\sigma_1$.

The intersections of the solutions with the axis $e_{jup} = 0$ correspond to bifurcations where the two branches, stable and unstable, merge into one periodic solution of the circular problem. The values $e = 0.73$ and $e = 0.45$ found for these bifurcations in the resonances 2/1 and 3/2, are the same known from the studies of periodic orbits of the restricted problem (see Hadjidemetriou, 1988). For each of the resonances 4/3, 5/4 and 6/5 two intersections are seen: the lower one is a bifurcation but those situated higher are only apparent and due to the graphic routines used.

Similar results were found for the resonance 3/1. In this case we have two bifurcations, one at the origin and the other at $e = 0.8$. These results deserve some comments. We first observe that the known and suspected 3/1 librators have eccentricities smaller than 0.6 (Nakai and Kinoshita, 1986; Klafke and Sato, 1989). The integrations made by Nakai and Kinoshita show that the eccentricities of (887) Alinda, (1915) Quetzálcoatl and (2608) Seneca may reach 0.8; in the case of Alinda it is also shown that, for the next 12,000 years, the angle $\varpi - \varpi_{jup}$ (or $\sigma_1 - \sigma$) will decrease from its present value of 85° to 71°, but it will increase again after that time. During the next 20,000 years this angle will vary of no more than 15° showing a situation very close to a secular resonance. However, this large variation in the eccentricity is not confirmed when the perturbations due to all known planets is considered (Milani *et al.*,1989). The variation in the next 20,000 years, according to Milani *et al.*, reaches only e=0.7 while values as high as 0.85 could have been reached 100,000 years ago, but before the capture of Alinda into a resonance 3/1.

The unstable corotation solution at the origin $e = e_{jup} = 0$, found in all resonances with $q > 1$, is on the origin of the eccentricity jumps found by Wisdom (1983). As he later showed (Wisdom, 1985), there is a chaotic region involving the small regular region of very low eccentricity, where the asteroid may be driven to eccentricities as large as 0.3. It is worth mentioning that Wisdom theories, as well as the one recently completed by Henrard and Caranicolas (1989), also show the high-eccentricity corotation zone but at a lower eccentricity (≈ 0.3), a result probably due to the inaccuracies of the classical Laplacian expansions used in their theories.

We mention that Morbidelli and Giorgilli also determined the corotation centers of some resonances. Their results, shown in Table I for $e_{jup} = 0.05$ agree with those of Ferraz-Mello (1990). They are also confirmed by Yoshikawa (1989) who mapped a twice averaged Hamiltonian. Yoshikawa's averaging is done over the mean anomaly of the asteroid motion and, then, over the argument of the perihelion (or, equivalently, the longitude of the ascending node, since his variables are chosen in such a way that Ω appears only through $\omega = \varpi - \Omega$). In this second averaging, the angle σ is kept constant and equal to the value it has at a libration center. The twice averaged Hamiltonian is then mapped into the plane $(e, \varpi - \varpi_{jup})$; the singular points correspond to corotation centers.

TABLE I. Corotation centers for $e_{jup} = 0.05$ (Morbidelli and Giorgilli, 1989).

Resonance	Eccentricity	$\varpi - \varpi_{JUP}$	Stability
3/2	0.49	0	unstable
	0.42	π	stable
2/1	0.76	π	unstable
	0.71	0	stable
5/2	0.014	0	unstable
	0.045	π	unstable
	0.27	0	stable
3/1	0.1	0	unstable
	0.82	0	unstable
	0.79	π	stable
7/2	0.026	0	unstable
4/1	0.011	0	unstable
	0.085	π	unstable
	0.45	0	stable

V. THE NON-REGULAR MOTIONS

The regular motions considered in this paper are oscillations about the stable periodic orbits of an abridged integrable version of the dynamical system representing the motion of an asteroid disturbed by Jupiter. In the phase space of this integrable system there are stable and unstable periodic orbits. For example, the apocentric branches, shown in figures 3 and 5, continue downwards, almost symmetrically to the pericentric branch, but as locus of unstable periodic solutions whose stable and unstable manifolds are the boundaries separating different classes of regular motions. When the full non-integrable dynamical system is considered, the thickness of these boundaries becomes finite and they become seat of chaotic motions. These chaotic zones are not the only ones and the improvement of the abridged integrable models shows a complex web of these manifolds (see Lemaître and Henrard, 1990). Anyway, when the asteroid moves far enough from Jupiter, the perturbations of these integrable models are small and we may be sure of finding regular motions around stable periodic orbits as the librations centers and the corotation centers.

Thus, the known *Hildas* seem to be moving in regions of regular motion, as far as the integrations are limited to 100,000 years and only the action of Jupiter is considered (Wisdom, 1987). We do not know what would happen if large time spans were considered. Indeed, the very nature of the Lyapunov test, used to decide on chaoticity or regularity, does not allow us to discard the possibility of finding a chaotic behaviour in future integrations extended over still larger times.

Some librating asteroids are, in fact, in a temporary capture at a resonance, moving inside the thick layers of chaoticity, and integrations over 100,000 are enough to reveal it. Also, the consideration of close approaches with Jupiter and the inner planets accelerates the excursions through the non-regular zones. One example is (887) Alinda whose crossings with the Earth will be responsible, in a short time, for a change from libration to circulation and, 10,000 years later, to a change back to libration (Milani *et al.* 1989). Another example is the asteroid (3552) 1983 SA, presently lying in the resonance 4/3 (see Figure 14) but, in fact, in a 60,000-year wandering path through the first-order resonances 3/2, 4/3, 5/4, 6/5 and 6/7 (this one exterior to the orbit of Jupiter).

VI. CONCLUSION

The introduction of new concepts for the expansion of the disturbing potential in the restricted problem allowed several authors to obtain results leading to a new insight into the regular solutions of the planar elliptic asteroidal problem averaged about a resonance with Jupiter. We mention the law of structure, relating the semi-major axis and the eccentricity of the periodic solutions called centers of libration, the stability and period of these solutions and the terms forced by the eccentricity of Jupiter. We also mention the corotation orbits – orbits where a mean-motion and a secular resonance happen simultaneously. In this case the eccentricities were determined as a function of the eccentricity of the disturbing planet (Jupiter).

It is worth emphasizing that almost all results described in this review were obtained without numerical integrations of the differential equations, and, in this sense, are general. They are, formally, analytical – even if numerical quadratures are necessary to obtain some functions; for example the coefficients R_j of eqn.(3) – and provide laws relating several kinematical parameters.

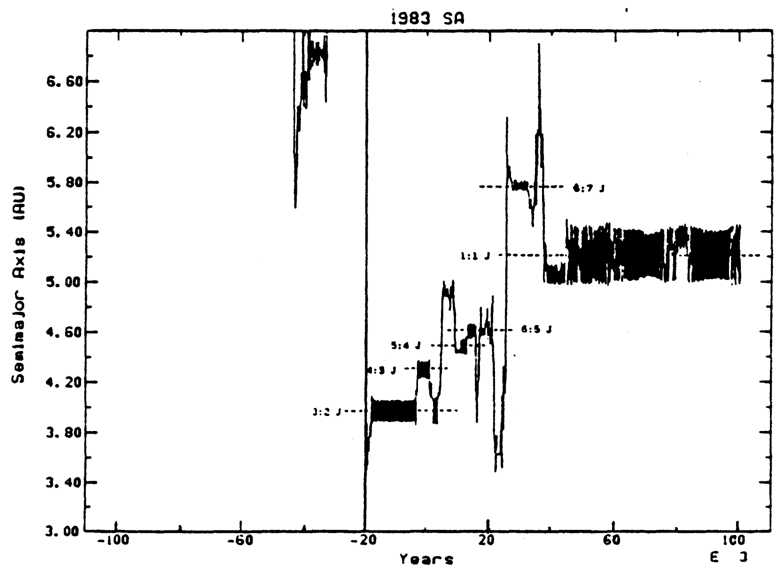


Figure 14. Variation of the semi-major axis of (3553) 1983 SA over 200,000 years (Milani *et al.*, 1989).

The results were compared to those obtained with other techniques, by other authors, showing good greement. The main quantitative differences where found when comparing to results obtained using the classical aplacian expansions which, as is well-known, fails to give a good representation of the disturbing function, even for noderate eccentricities. An important limitation of the results obtained so far lies on the fact that the only planet onsidered is Jupiter and even this planet is taken in a fixed orbit. In fact, the motion of the perihelion of Jupiter affects he results by changing the scale of the figures obtained along the vertical axis. For instance the values of the eccentricities f the corotations centers are smaller when the perihelion moves directly. Thus, as the motion of the perihelion of Jupiter aries from -10 to +10 arcseconds per year (Knezevic, 1986; Laskar, 1988), with a relatively short period (40,000 years), he actual motions may have features still not shown in the models considered.

cknowledgements. The preparation of this paper was sponsored by the Research Foundation of the State of São Paulo FAPESP).

EFERENCES

erraz-Mello, S.: 1987a, "Averaging the elliptic asteroidal problem near a first- order resonance", *Astron. J.* **94**, 208-212.

erraz-Mello, S.: 1987b, "Expansion of the disturbing force-function for the study of high-eccentricity librations", *Astron. Astrophys.* **183**, 397-402.

erraz-Mello, S.: 1988, "The high-eccentricity libration of the Hildas". *Astron. J.* **96**, 400-408.

erraz-Mello, S.: 1989a, "A semi-numerical expansion of the averaged disturbing function for some very-high-eccentricity orbits", *Celest. Mech.* **45**, 65-68.

erraz-Mello, S.: 1989b, "Averaging Hamiltonian Systems", in D.Benest and Cl.Froeschlé (eds.) *C.R. de l'Ecole d'Astrophysique de Printemps*, Soc. Franc. Spec. Astron. (in press)

erraz-Mello, S.: 1990, "A theory of planar planetary corotations" (in preparation).

erraz-Mello, S., Sato,M.: 1989, "A very-high-eccentricity asymmetric expansion of the disturbing function near resonances of any order", *Astron. Astrophys.* (in press).

roeschlé, Cl., Scholl,H.: 1976, "On the dynamical topology of the Kirkwood gaps", *Astron. Astrophys.* **48**, 389-393.

roeschlé, Cl., Scholl,H.: 1981, "The stochasticity of peculiar orbits in the 2:1 Kirkwood gap", *Astron. Astrophys.* **93**, 62-66.

iffen, R.: 1973, "A study of commensurable motion in the asteroidal belt", *Astron. Astrophys.* **23**, 387-403.

ireenberg, R., Franklin, F.: 1975, "Coupled librations in the motion of asteroids near the 2:1 resonance" *Mon. Not. R. Astron. Soc.* **173**, 1-8.

Hadjidemetriou, J.D.: 1988, "Periodic orbits of the planetary type and their stability" *Celest.Mech.* **43**, 371-390.
Henrard, J., Caranicolas, N.: 1989, "Motion near the 3/1 resonance of the planar elliptic restricted three body problem" *Celest.Mech.* (in press)
Klafke, J.C., Sato., M.: 1989, "Asteroidal Motion near the 3/1 commensurability", in R.Vieira Martins *et al.* (eds.)*Orbital Dynamics of Natural and Artificial Objects*, Observ. Nacional, Rio de Janeiro, pp.23-37.
Knezevic, Z.: 1986, "Secular variation of major planets. Orbital elements" *Celest. Mech.* **38**, 123-138.
Laskar, J.: 1988, "Secular evolution of the Solar System over 10 million years" *Astron. Astrophys.* **198**, 341-362.
Lemaître, A., Henrard, J.: 1988, "The 3/2 resonance" *Celest. Mech.* **43**, 91-98.
Lemaître, A., Henrard, J.: 1990, "On the origin of cahotic behaviour in the 2/1 Kirkwood gap" *Icarus* (in press).
Milani, A., Carpino, M., Hahn, G., Nobili, A.M.: 1989, "Project Spaceguard: Dynamics of planet-crossing asteroids. Classes of Orbital Behaviour" *Icarus* **78**, 212-269.
Morbidelli, A., Giorgilli, A.: 1989, "On the Dynamics of the Asteroidal Belt. *Celest. Mech.* (in press)
Nakai, H., Kinoshita, H.: 1986, "The motion of numbered asteroids in resonance regions" *Proc.19th. Symposium on Celestial Mechanics*, Akashi(Japan), pp.1-9.
Poincaré, H.: 1902, "Sur les problèmes du type d'Hécube", *Bull. Astron.* **19**, 289- 310.
Schubart, J.: 1968, "Long-period effects in the motion of Hilda-type planets" *Astron. J.* **73**, 99-103.
Tsuchida, M.: 1990, "Libration motion of asteroid (279) Thule", *Rev. Mexic. Astron. Astrophys.* (this issue).
Vienne, A.: 1987, "Mouvement des petites planètes du groupe Hilda", *D.E.A. d'Astronomie*, Observ. Paris, Paris.
Williams, J.G., Faulkner, J.: 1981, "The positions of secular resonance surfaces" *Icarus* **46**, 390-399.
Wisdom, J.: 1983, "Chaotic behaviour and the origin of the 3/1 Kirkwood gap" *Icarus* **56**, 51-74.
Wisdom, J.: 1985, "A perturbative treatment of motion near the 3/1 commensurability" *Icarus* **63**, 272-289.
Wisdom, J.: 1987, "Chaotic dynamics in the Solar System" *Icarus* **72**, 241-275.
Yoshikawa, M : 1989, "A survey of the motions of asteroids in the commensurability with Jupiter" *Astron. Astrophys.* **213**, 436-458.

Sylvio Ferraz-Mello: Universidade de São Paulo, Instituto Astronômico e Geofísico, Caixa Postal 30627, 01051 São Paulo, Brasil.