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NON-MINIMALLY COUPLED THEORIES: ANISOTROPIC COSMOLOGIES THROUGH HYPEREXTENDED GRAVITY

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RESUMEN

Se analizan cosmologías anisotrópicas en el marco de teorías escalares-tensoriales de gravitación. Se generalizan métodos algorítmicos previos con el objeto de poder estudiar cualquier tipo de teoría acoplada no-mínimamente.

ABSTRACT

Anisotropic cosmologies in general scalar-tensor gravitation —including non-minimally couplings— are considered. Previous algorithm methods are generalized in order to admit the study of any non-minimally coupled theory.

Key words: **COSMOLOGY – THEORY — GRAVITATION**

1. INTRODUCTION

Scalar-tensor gravitation have proved to be a useful tool in the understanding of early universe models. The first and best known case of such theories is Brans-Dicke (Brans & Dicke 1961, BD) gravity, in which there is a coupling function $w(\phi)$ equal to a constant; ϕ being a dynamical field related with the previous gravitational constant. More general theories with other couplings have also been studied (Bergmann 1968; Nortvedt 1970; Wagoner 1970). The interest on these theories has been recently rekindled by inflationary scenarios (Fakir & Unruh 1990; La & Steinhardt 1989; Steinhardt & Ascetta 1990) and fundamental theories which seek to incorporate gravity with other forces of nature. Particularly, in string theories, a dilaton field coupled to curvature appears in the low energy effective action (Fradkin & Tseytlin 1985; Callan et al. 1985; Lovelock 1985). When scalar-tensor gravitation is concerned, one is interested also in the cosmological models it leads. Observational constraints, mainly coming from the weak field tests (Will 1981) and nucleosynthesis (Serna, Dominguez-Tenreiro, & Yepes 1992; Casas, García-Bellido, & Quirós 1992; Torres 1995), put several bounds upon the couplings. In any case, in order to evaluate the cosmological scenario and to test the predictable force of any theory, it is desirable to have exact analytical solutions of the field equations. But it was only a few years ago, that Barrow (1993), Barrow & Mimoso (1994), and Mimoso & Wands (1995a) derived algebraic-numerical methods that allow exact Friedmann-Robertson-Walker (FRW) solutions to be found in models with matter content in the form of a barotropic fluid for any kind of coupling $w(\phi)$.

However, scalar-tensor theories can be formulated in two different ways depending on the choice of the basic action. These two possible choices are the BD one, in which there is an arbitrary function in the kinetic term of the scalar, and the one which admit an arbitrary function multiplying the curvature while mantaining a common kinetic term, for the theories known as non-minimally coupled (NMC). Via a field redefinition one can establish the equivalence between these choices. But if the functions involved are not analytically invertible as, for instance, in the hyperextended inflationary scenario (Liddle & Wands 1992), we are left without any analytical algorithm technique in order to get the solutions of the system. Very recently, we have presented a study on the full lagrangian density for the field, which involves, in the more general case, two free functions (Torres & Vucetich 1996). This lagrangian reads

$$L = 16\pi L_m - \frac{w(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} + G(\phi)^{-1}R \ , \quad (1)$$

where, as usual, L_m refers to the matter lagrangian density and R is the curvature scalar. Each possible choice of the action may be reproduced by a convenient selection of G and/or w. We have called hyperextended scalar-tensor gravitation (HSTG) to the theories of gravity that this lagrangian leads, because of the similarity with the inflationary model. Similar algorithm methods of massless scalar fields, developed by Mimoso & Wands (1995a), were applied to this general approach. Those methods allowed to compute exact FRW analytical solutions in vacuum, or with matter content consisting of radiation or stiff fluids for any choice of G and w simultaneously (Torres & Vucetich 1996). That includes the cases of NMC, where solutions are scarce (Capozziello & de Ritis 1994). This approach was also used in the search for slow-roll solutions for non-minimally coupled theories (Torres 1997).

Anisotropic homogeneous cosmological models are being intensively studied for quite a long time (MacCallum 1979). In particular, within scalar-tensor gravitation, the analysis of anisotropic cosmologies could reveal different behavior when compared with Einstein General Relativity near the singularity (Ruban & Finkelstein 1972), or in the inflationary epoch (Pimentel 1989). Processes of isotropization of Brans-Dicke Bianchi-type solutions are also of current interest (Chauvet & Cervantes-Cota 1995).

The aim of this work is to explore, within the general lagrangian density of HSTG, the case of some simple anisotropic cosmological models. In particular, we are interested in examining the possibility of extending previously derived results for anisotropic models in BD gravity to this general approach; allowing, for instance, to compute also analytical anisotropic solutions for non-minimally coupled theories. Throughout this work, then, we draw on the results on massless fields in anisotropic universes (Mimoso & Wands 1995b) and in the properties of the lagrangian (equation 1) when considered in the Einstein frame. A recent study of the cosmological conformal equivalence between the Jordan and the Einstein frame was presented by Capozziello, de Ritis, & Marino (1997). The leading idea of this equivalence is to make a transformation of the dynamical fields so as to simplify the equations, i.e., to recover their Einstein form. Once the solution is obtained in that simplified picture one can go back and translate the results to the physical metric in the Jordan frame. This procedure allows us to adapt General Relativity solutions to more general scalar-tensor theories.

2. EXTENDING THE FORMALISM

The field equations of HSTG are[1]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G(\phi)\left[8\pi T_{\mu\nu} + \frac{\omega}{\phi}\phi_{,\mu}\,\phi^{,\mu}\right.$$

$$\left. - \frac{\omega}{2\phi}\phi_{,\alpha}\,\phi^{,\alpha}g_{\mu\nu} + (G^{-1})_{,\mu;\nu} - g_{\mu\nu}\Box(G^{-1})\right] , \quad (2)$$

$$\dot{\phi}^2\left[\frac{1}{\phi}\frac{d\omega}{d\phi} - \frac{\omega}{\phi^2} + G\,\frac{dG^{-1}}{d\phi}\frac{\omega}{\phi}\right] + \frac{2\omega}{\phi}\,\Box\phi +$$

$$3G\,\frac{dG^{-1}}{d\phi}\Box(G^{-1}) - G\,\frac{dG^{-1}}{d\phi}8\pi T = 0. \quad (3)$$

It is very important to remark that the usual relation $T^{\mu\nu}{}_{;\nu} = 0$ establishing the conservation laws (in the meaning of GR) of the matter fields holds true. Note that when $G = 1/\phi$, these equations reduce to the common BD ones. The Einstein frame was introduced by Dicke (1962), when working in Brans-Dicke gravity, by defining a conformal transformation of the form

$$\tilde{g}_{ab} = G_0\phi g_{ab} \ , \quad (4)$$

where G_0 is an arbitrary constant which becomes the gravitational constant in the transformed frame. In a similar fashion, we introduce

$$\tilde{g}_{ab} = G_0 G(\phi)^{-1}g_{ab} \ . \quad (5)$$

Using the relation between the curvature scalars of the common –Jordan– frame and the transformed –Einstein– frame, given for all conformal transformations by Synge (1960), we get for the action

$$S_{EF} = \frac{1}{16\pi}$$

$$\int\sqrt{-\tilde{g}}\ \left[\frac{1}{G_0}\tilde{R} - \tilde{g}^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}\frac{3}{2G_0}\alpha + 16\pi\tilde{L}_m\right] \ , \quad (6)$$

where we have defined $\tilde{L}_m = L_m/(G_0G(\phi)^{-1})^2$. It is now possible to define a new scalar field ψ by

$$d\psi = \sqrt{\frac{3}{16\pi G_0}\alpha}\frac{d\phi}{\phi} \ , \quad (7)$$

[1] Equation (3) is obtained assuming that ϕ depends only on time.

such that

$$S_{EF} = \frac{1}{16\pi}$$

$$\int \sqrt{-\tilde{g}} \left[\frac{1}{G_0} \tilde{R} + 16\pi (\tilde{L}_m - \frac{1}{2} \psi_{,a} \psi^{,a}) \right] . \quad (8)$$

Thus, we recover the Einstein action with a stress energy tensor given by the sum of two contributions, matter, and scalar fields. It is worth noting that the scalar field ψ is proportional to the variable Y which was introduced to solve the problem of FRW models in Torres & Vucetich (1996). The recovery of the Einstein action —and the field equations derived from it— does, however, have a cost. That is, a non-independently conserved stress-energy tensor for matter, which in the Einstein frame behaves in agreement with

$$\tilde{T}_{ab}{}^{;a} = \frac{1}{2} \sqrt{\frac{16\pi G_0}{3}} \frac{1}{\alpha} \tilde{T} \psi_{;b} \left(\frac{\phi}{G^{-1}} \frac{dG^{-1}}{d\phi} \right) . \quad (9)$$

So, in any case in which $\tilde{T} \neq 0$, only the total stress-energy tensor will be conserved, i.e., $(\tilde{T}_{ab} + \tilde{T}^{\psi}_{ab})^{;a} = 0$; where $\tilde{T}^{\psi}_{ab}$ stands for the stress-energy tensor related with the field, which is given by

$$\left(\tilde{T}^{\psi}_{ab} \right)^{;a} = \left(\psi_{,a} \psi_{,b} - \frac{1}{2} \psi_{,c} \psi^{,c} \tilde{g}_{ab} \right) . \quad (10)$$

Equation (10) and cosmological assumptions for the field show that, in the Einstein frame, it behaves as a stiff fluid with density and pressure given by

$$\tilde{\rho}^{\psi} = \tilde{p}^{\psi} = \frac{1}{2} \left(\frac{d\psi}{d\tilde{t}} \right)^2 . \quad (11)$$

Due to the exact reproduction of the the Einstein field equations, the common results of general relativity will apply. We shall take into account matter given by a barotropic fluid, particularly in the cases of stiff fluids, radiation, or both. The case of a dust fluid plus radiation in scalar-tensor theories was analyzed by Torres & Helmi (1996), while some cases of imperfect fluids were studied by Pimentel (1994). Following Raychaudhuri (1979), and considering those models in which the velocity of matter is parallel to the unit normal to the spatial hypersurfaces (a geodesic time-like vector t^a) it is possible to write the Einstein field equations —the HSTG equations in the Einstein frame— in the form of a constraint

$$\tilde{\theta}^2 = 24\pi G_0 (\tilde{\rho} + \tilde{\rho}^{\psi}) + 3\tilde{\sigma}^2 - \frac{3}{2} \, {}^3\tilde{R} , \quad (12)$$

plus the Raychudhuri equation

$$\frac{d\tilde{\theta}}{d\tilde{t}} + \frac{1}{3} \tilde{\theta}^2 = -4\pi G_0 [\tilde{\rho} + \tilde{\rho}^{\psi} + 3(\tilde{p} + \tilde{p}^{\psi})] . \quad (13)$$

Here, we have introduced the expansion $\tilde{\theta}$, the shear $\tilde{\sigma}$ and the curvature scalar of hypersurface of homogeneity ${}^3\tilde{R}$; all them, in the Einstein frame. The transformed quantities are

$$\tilde{\rho} = \frac{\rho}{(G_0 G(\phi)^{-1})^2} , \quad \tilde{p} = \frac{p}{(G_0 G(\phi)^{-1})^2} ,$$

$$\tilde{\sigma}^2 = \frac{\sigma^2}{(G_0 G(\phi)^{-1})} , \quad d\tilde{t}^2 = \left(G_0 G(\phi)^{-1} \right) dt^2 , \quad (14)$$

we also introduce a volume factor $\tilde{V} = \left(G_0 G(\phi)^{-1} \right)^{3/2} V$, with V such that $\theta = dV/V dt$.

Having available both conservation laws, for matter in the Jordan frame and for matter plus field in the Einstein frame, it is possible to derive the corresponding energy densities and pressures. They are

$$\tilde{\rho} = \frac{3}{8\pi G_0} \left(\frac{\Gamma}{\tilde{V}^{4/3}} + \frac{M G_0 G(\phi)^{-1}}{\tilde{V}^2} \right) , \quad (15)$$

$$\tilde{p} = \frac{3}{8\pi G_0} \left(\frac{\Gamma}{3\tilde{V}^{4/3}} + \frac{M G_0 G(\phi)^{-1}}{\tilde{V}^2} \right) , \quad (16)$$

$$\tilde{\rho}^{\psi} = \tilde{p}^{\psi} = \frac{3}{8\pi G_0} \left(\frac{A^2 - 4M G_0 G(\phi)^{-1}}{4\tilde{V}^2} \right) . \quad (17)$$

Here, Γ is related with the presence of radiation and M with the presence of a stiff fluid. Both, Γ and M, are positive constants. When a stiff fluid is present, it is possible to define a new field χ, minimally coupled to the metric, such that its energy density is the sum of the energy density of the scalar ψ and the matter content. This field χ can be related to ϕ by (Mimoso & Wands 1995b; Torres & Vucetich 1996)

$$\sqrt{\frac{16\pi G_0}{3}} \chi(\phi) = A \int \frac{d\tilde{t}}{\tilde{V}} = \pm$$

$$\int \sqrt{\frac{A^2}{A^2 - 4M G_0 G(\phi)^{-1}}} \sqrt{\alpha} \, \frac{d\phi}{\phi} . \quad (18)$$

We see that, defining a new set of functions (G, w) by

$$\alpha_{vac} = \frac{A^2}{A^2 - 4MG_0G(\phi)^{-1}}\alpha \qquad (19)$$

the effect of stiff fluid matter is equivalent to a new lagrangian with no matter content. That change is possible because the stiff fluid modifies the dependence of χ with ϕ, work that in vacuum models is accomplished by w and G. That was exactly what happened with BD models, where only w existed. Here also, the field equations become simplest with the use of the variable $X = \left(G_0G(\phi)^{-1}\right)a^2 = \tilde{a}^2$ and the conformal time $d\eta = adt$ (Torres & Vucetich 1996). Let us treat in this formalism and as a first example, what happens with Bianchi I universes.

3. A SIMPLE EXAMPLE: BIANCHI I UNIVERSE

The metric of Bianchi I models is

$$ds^2 = dt^2 - a_1(t)^2dx^2 - a_2(t)^2dy^2 - a_3(t)^2dz^2 \, . \qquad (20)$$

Here, the Jordan frame expansion is given in terms of an averaged scale factor $a^3 = a_1a_2a_3 = V$ in the form $\theta = 3da/adt$. The spatial curvature is null and the metric reduces to the flat FRW one for the case in which $a_1 = a_2 = a_3$. The general relativistic result for the shear holds in the Einstein frame; i.e., $\tilde{\sigma}^2 = 3\Sigma^2/4\tilde{a}^6$, where Σ is a constant and $\tilde{a}^3 = \tilde{V}$. Using the expressions for the energy densities, it is possible to obtain the constraint equation in the variable X. For matter given in the form of non-interacting stiff and radiation fluids in the Jordan frame, it is

$$X'^2 = A^2 + \Sigma^2 + 4\Gamma X \;, \qquad (21)$$

where as before $X = \tilde{a}^2$. It has exactly the same form that the equation obtained in (Mimoso & Wands 1995b) and thus it admits the same solution (equation [105] of that paper)[2]. Equation (21) is in fact a general relativistic result, valid here because of the Einstein frame (Ruban 1978). It is important to stress that, although the equation and its solution are the same, the meaning of the variables are different and that the study of the more general case of lagrangian (equation 1) is now allowed. The solution of (equation 21) shows shear-dominated evolution at early times ($\tilde{a}^2 \to 0$) and radiation-dominated evolution when the averaged scale factor tends to infinity ($\tilde{a}^2 \to \infty$). Note that these results do not depend on the particular form of G nor that of w. The specification of X, the shear, and each one of the scale factors of the metric —which can be obtained from the general relativistic results in the Einstein frame— describe the full evolution of the system. To go back to the Jordan frame, we have to know $\phi(\eta)$ by inverting (equation 18) and obtaining $\phi(\chi)$. Afterwards, we use our knowledge of $X(\eta)$ to get $\chi(\eta)$. Both operations yield finally, $\phi(\eta)$. From equation (18) we see that for an equal functional form of α_{vac}, an equal solution for the field ϕ is obtained. Indeed, it means that if a solution for ϕ in a particular BD gravity with $w = w_{vac}$ is known, it is also a solution for the set of theories given by $\alpha_{vac} = (2w_{vac} + 3)/3$. Thus, it is possible to define α_{vac} as a classifier of equivalence sets. Within each one of these sets, the definition of X allows us to obtain the particular behavior of the averaged scale factor. The functional form of the metric scale factors does not depend on w or G in the Einstein frame, while it does in the Jordan frame. It is in this frame where a particular dependence of G on ϕ discriminate among different behaviors inside an equivalence set. As was the case in BD gravity, some conclusions may be obtained without the specification of the functions involved in the lagrangian density. In the presence of a stiff fluid in the Jordan frame, a bounce will occur when

$$\frac{da}{dt} = \frac{1}{2}\left(\frac{X'}{X} - G(G^{-1})'\right) = 0 \;, \qquad (22)$$

which requires

$$(A^2 + \Sigma^2)\alpha = (A^2 - 4MG_0G^{-1})\left(\frac{\phi}{G}\frac{dG}{d\phi}\right)^2 \, . \qquad (23)$$

It may be also seen that, if G^{-1} vanishes faster than X^3, an anisotropic initial singularity in the Einstein frame becomes isotropic in the Jordan frame. Note that the condition for the bounce is incompatible with a vanishing G^{-1} and X. The possibility of having a finite expansion in this situation was analized for BD gravity (Mimoso & Wands 1995b). As the formalism here developed is based on the exact reproduction of the equations, via a suitable change of variables, the analysis provided in the BD case will continue to hold. For the sake of conciseness we do not go through it in depth but refer the interested reader to Mimoso & Wands (1995b).

4. CONCLUDING REMARKS

The above analysis implies that the algorithm method developed by Mimoso and Wands for Brans-Dicke theory is capable to deal also with Bianchi I models in general hyperextended theories. As a matter of fact, one can —at this stage— observe that the same applies to other models, such as Bianchi V or III. Those models are studied by Ruban (1978) for General Relativity and by Mimoso & Wands (1995b) in BD gravity, and to analytically obtain NMC solu-

[2] $X = \tilde{a}^2 = |\eta - \eta_0|\left(\sqrt{A^2 + \Sigma^2} + \Gamma|\eta - \eta_0|\right)$.

tions in this same way it is only necessary to translate that analysis to the variables X, ψ and to equations (18) and (19) of this work.

It is important to stress that the formalism presented here as well as in Torres & Vucetich (1996) does not mean that BD and NMC theories are not equivalent via field transformations; something that has been known for a long time. On the contrary, it is to be noticed that whenever non-invertible functions are used, these transformations cannot be performed analytically and then, the algorithm procedures that allow us to find solutions cannot be applied. It is in these cases that a HSTG scheme is convenient.

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